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MY FIFTY YEARS OF MATHEMATICS.

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BY H. R. HASSÉ.

THE title of my address this morning is " My Fifty Years of Mathematics ". The use of the possessive pronoun is deliberate, and it might have been better if I had used it again so that the title would then have read " My Fifty Years of My Mathematics ", for it is obvious that no one would attempt to cover the whole of mathematics over any length of time, however short the period. What I propose to do is to sketch very inadequately the changes, in range and in character, of the teaching of mathematics to University students during the time in which I have been a student and a teacher. Those of you who are teachers in schools will be able to judge, better than I can, to what extent these changes have influenced the teaching in the higher forms in schools. In particular, I shall not attempt to carry you on to the boundaries of existing knowledge in any direction. My reaction to most of the work of the last twenty years cannot be better expressed than in the words of a late Victorian poet : " If this young man expresses himself in terms too deep for me, why what a very singularly deep young man this deep young man must be."

It is over fifty years since I entered Manchester University as a student ; in those days there was no Higher School Certificate examination, and the entrance qualification to a University was, in general, the passing of the matriculation examination of the particular University. It was only in a strictly limited number of cases that pupils remained at school after the age of 16 or 17, and so were enabled to study for entrance scholarships at Oxford and Cambridge.

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Looking back over my undergraduate study at Manchester and later at Cambridge, the teaching was characterised by the absence of what we now know as analysis, by the predominance of a geometrical outlook, and, in applied mathematics, by the divorce between theory and experiment. Before going any further, I must explain that this statement must not be taken as meaning that all students were ignorant of the Dedekind theory of sections, or that they knew nothing of experimental physics. The *real* mathematician —I am avoiding the use of the term " first-class ", which at that stage in one's life savours too much of the examination system—will survive the effects of any teaching and of any syllabus, and we all know that there has been a steady stream of real mathematicians from the universities for a much longer period than the past fifty years.

The textbooks used by the average student in the first years of this century, many of which are still useful as repositories of facts and formulae, almost without exception came to grief when dealing with infinite processes. The usual proof at that time of the fundamental theorem of inequalities, that the arithmetic mean A of n positive numbers $a_1, a_2, \ldots, a_n$, is greater than their geometric mean G, was as follows. Taking any two of the numbers, say a_1 and a_2, and replacing each of them by $\frac{1}{2}(a_1+a_2)$, A remains unaltered and G is increased, since $\frac{1}{4}(a_1+a_2)^2 > a_1a_2$. So long, therefore, as G contains two unequal factors it can be increased in this way, and hence G is greatest when its factors are all equal and then $A=G$, so that originally $A>G$. But, except when n is of the form 2^m, the process is not completed in a finite number of steps, and the proof is therefore no longer algebraic, but belongs to the domain of analysis. The above proof can easily be repaired to deal with any n, and is then perhaps the most elementary of the many proofs of this theorem, and there are two algebraic proofs, based on ideas similar to those of the above defective proof, which are quite satisfactory.

Again, the exposition of the convergence and divergence of infinite series was often at fault, owing to the fact that such series can also oscillate ; when this was ignored, the result was an incorrect treatment of the behaviour of alternating series of positive and negative terms. To take a third example, the evaluation of infinite integrals by the process of differentiating under the integral sign could not be put on a proper basis until the idea of uniform convergence had found its way into textbooks on the integral calculus.

To the average student pure geometry at that time can have meant little more than the use of one or other of the processes of inversion, reciprocation or projection for the solution of problems. As an extreme example of how examinations can be used to perpetuate a restricted point of view, candidates were not allowed to use the methods of the calculus or of analytical geometry in the only complete paper set on geometry in the Cambridge Tripos examination. The existence of different geometries and the relationships between them can therefore have been known only to those few students who were so entranced by the beauty and generality of the above-mentioned processes that they felt themselves compelled to pursue their studies beyond what was required for examination purposes.

That there was, in spite of what has just been said, a definite geometrical outlook, particularly in applied mathematics, is evident from the fact that the syllabus for the Tripos contained the first three sections of Newton's *Principia*, dealing respectively with limits, central forces, and the inverse square law of force. The requirements of the student in this respect were catered for by a book, in the preface to which the author states : " I have also endeavoured to give assistance to the student who is engaged in the study of the higher branches of mathematics by presenting in a geometrical form several of the processes employed in the differential and integral calculus and

in the analytical investigations in dynamics." Nothing better can prove the existence of a geometrical outlook than this statement, which is so completely at variance with the ideas of today. It is true that the book I am referring to was first published in 1863, but these three sections of Newton's *Principia* remained in the syllabus until 1907, and lectures on them were given as part of the course of study, though it does not follow that much attention was paid to this last survival of Newtonian methods ; probably the contrary was the case as regards the large majority of students.

The books on applied mathematics of that period are full of geometrical proofs, interpretations, and illustrations of the theorems and methods contained in them ; the same is also true, though to a smaller extent, of Lamb's standard books on *Statics*, *Dynamics* and *Higher Mechanics*. As an example I take the geometrical construction for the two parabolic paths from one given point A to another B, with a given speed of projection from A, in the motion of a particle under gravity, and the corresponding constructions for the two ellipses in the case of motion under an attractive force varying as the distance from a fixed point and as the inverse square of the distance respectively. These constructions lead to the determination of the enveloping parabola and ellipses, and are all to be found in Lamb's *Dynamics*. If we examine recent textbooks, it will appear that there is a tendency to give the geometrical construction only for the case of motion under gravity, and in one book published last year there is no mention even of this construction. Some may deplore the absence of this type of geometrical construction, but it is a sign of the gradual replacement of a geometrical outlook by a purely analytical point of view which has been steadily taking place during the last fifty years.

I now come to the outstanding event of this period—the reform of the Mathematical Tripos in 1907—which has had the greatest influence on mathematical teaching in Great Britain, and which I remember well as I was at Cambridge during the discussions which took place then ; unfortunately, not being eligible then to take the M.A. degree, I could not register a vote in favour of the proposals.

The Mathematical Tripos has naturally undergone many changes, both as regards regulations and syllabus, at various times in its long history, as mathematical knowledge increased, as developments in various branches of mathematics brought them into prominence, and as the schools were able to extend the range of mathematical teaching. In its form in 1907, it dated from 1882, when the last change had taken place ; it was realised then that very little could be done unless at the same time the question of the order of merit in the published class list was dealt with, but those who advocated the changes then proposed fought shy of even considering the abolition of the order of merit, which they knew would cause a split in their ranks and be likely to result in there being no changes at all. Their successors in 1906–07 were much bolder, and more united in their policy. Early in 1906 the Special Board of Mathematics proposed splitting the examination into two parts, which are essentially Part I and Part II of the Tripos as it is at present, and abolishing the order of merit. In the report of the Board, it is argued that, as regards the order of merit, " the ablest students, being thus relieved from the necessity of competing for places in the examination, would be able to spend a considerable portion of their time during the first three years on advanced work, whilst it is hoped that other students of somewhat less ability would be encouraged to spend part of their time in this manner ".

On the proposals in general, the Report states, of the examination : " The work of at least half of the candidates has not been found to be satisfactory

in quality. One cause of this is believed to be that such candidates are not encouraged to spend enough time and thought in attaining clearness as regards general principles, the mere manipulation of symbols occupying too much of their attention "; and "the questions set in mathematical physics are often of such a character as to maintain and emphasise the divergence which has obtained within the university between the study of physics in its abstract and on its experimental side, to the great disadvantage of students of the subject".

The files of *The Times* contain many letters and articles on the controversy which these proposals raised. It centred round the question of the prestige of the senior wrangler and the traditions associated with the order of merit, which went back at least 125 years. To quote from two of these letters, Hobson, then chairman of the Special Board, wrote: "In the opinion of many of us it is a fatal objection to the present examination that it induces our students to work, not in the best way, at wrong parts of the subject, for the whole length of time. The examination is, after all, only a test at the end of the course; if we can arrange it that it may induce our students to work in the best way, at the right branches, and for the right length of time, we need have no fear for the prestige of our school of mathematics"; and Forsyth: "Many who speak with approval of the Tripos as it was 40 or 50 years ago might be astonished to find how technical it has become in its later period, and how skill in an ordered technique has gradually been developed while the necessity for reasoned thought has diminished."

The report of the Special Board was voted on by the Senate in November 1906, with the following result:

Institution of the New Part I examination: for, 203; against, 183,

Abolition of the order of merit and institution of the new Part II examination: for, 203; against, 169,

showing 53% and 55% in favour of the proposals. This was hardly a satisfactory result, and both sides proceeded to try and interpret it to their own advantage, thereby showing that mathematicians are very human individuals after all. Those in favour of the proposals could not resist the temptation to make deductions which were more in their favour than the voting indicated. They argued that only the votes of resident members should count, and that of these 58% and 60% were cast for the two proposals respectively. To this the opposition replied that what was really important was the fact that the majority of the college lecturers in mathematics were opposed to the scheme, that the mathematical staffs of ten colleges opposed it, while those of six were in its favour, and that only 15 out of 25 of the members of the Special Board signed the report. One is reminded strongly of the post-mortem examinations held on the results of the last general election. Both sides now appealed to *all* members of Senate, resident and non-resident, and conducted a canvass of individual members, for a final trial of strength. The Great Northern Railway of those days arranged to provide special accommodation on four of their ordinary trains from London, which they were prepared to run in duplicate if necessary, and also to run a special late train from Cambridge to London. Presumably this train would allow, since voting closed at 7 p.m., non-residents to dine in college after their labours. The voting, which took place in February 1907, gave almost the same results as before, *e.g.* that for the institution of the new Part II examination with the abolition of the order of merit was 780 for and 638 against, again 55% in favour.

Thus was brought about the reform of the Mathematical Tripos, an event which has profoundly altered mathematical teaching in both universities and schools. This marked the passage from the "problem age" to the "example age"; in the former a highly developed technique was demanded of the

student, while in the latter an understanding of the theorem which it is intended to exemplify is required. Some may hold the view that we have not gone far enough in this last direction, and that tradition, fortified by the examination system, still exerts too strong an influence. Others may feel that we have gone too far and that, as has been expressed by someone, there is a danger that now we " read mathematics instead of learn mathematics ". We know only too well in these days that what is desirable is a proper balance between conflicting opinions and how difficult it is to attain it. In applied mathematics, where there are few principles and few important theorems, as compared with pure mathematics, it is hardly possible to avoid setting occasionally in examinations a problem by itself, that is, without accompanying theorem. I suggest that in this case an examiner should ask himself what principle or what theorem the proposed question is intended to exemplify, and then judge whether the question is worth setting at all ; if this procedure lessens ordeal by examination, so much the better.

The stage having been set, there was no lack of actors to carry on with the new type of drama under the new management. It was fortunate indeed that in analysis Bromwich and Hardy had just written books, both published in 1908, which were so successful that they are still essential for any teacher. The reviews in the *Mathematical Gazette* show how they met in different ways and for different purposes the defects in the older mathematical textbooks and started the teaching of analysis in Great Britain on the right lines. Of Bromwich's *Infinite Series* the reviewer remarks : " I began by explaining the poverty of English writers on series ; let me end by expressing the opinion that Mr. Bromwich's book is decidedly more complete, and in other ways more satisfactory than any of the continental treatises with which it is at all comparable " ; and on Hardy's *Pure Mathematics* the same reviewer writes : " When Mr. Hardy sets out to prove something, then, unlike the writers of too many widely read textbooks, he really does prove it " ; and " I for one shall hope to avoid in the future the many weary hours that have usually to be spent in convincing university students that ' proofs ' which they have laboriously learnt at school are little better than nonsense ". Their successors have provided the student at school and at the university with books both in pure and applied mathematics, which cover the other branches of mathematics and, together with the best of the " popular " books on the subject, give both the wider and deeper outlook which, in the early years of the century, could hardly be found save in continental books.

I now turn to the consideration of other events which have had an influence on mathematical teaching in the last fifty years. What I have to say must of necessity be very fragmentary in character, as I can make no claim to be an expert on any one of the items I shall mention, unless it be that I satisfy the definition of a military expert which I read recently, as one who is thousands of miles away from the base of operations. It would be strange, indeed, if the impact of two World Wars had not had a lasting effect on mathematics, as it undoubtedly has had on all other sciences, and on the scientific outlook of mankind in the whole world. The First World War gave an impetus to the study and use of numerical methods in the solution of the many problems which could not then be, and in most cases have not yet been, solved by analytical methods. Investigations of this character proceeded steadily, and increased in number, between the two wars to such an extent that numerical methods have almost entirely taken the place of graphical, *i.e.* geometrical, methods which did not satisfy the requirements (1) of being sufficiently accurate, (2) of being adaptable to the determination of the effects of small changes in the data of the problem, and (3) of being self-checking or, when necessary, of being checked independently. As an example which will be

familiar to everyone, " graphical statics " has practically disappeared as a chapter in books on statics, and does not hold the dominant place in engineering practice that it had fifty years ago. The American publication, *Mathematical Reviews*, started in 1940, contains a section labelled " Graphical and Numerical Methods ", though, judging by its contents, the word " graphical " should have been omitted. The section deals with algebraic equations, linear equations and determinants, differential and integral equations, harmonic analysis, instruments and machines, computation of series, and nomography, of which only the last named can be said to involve graphical representation. Our own *Science Abstracts* now contains a few pages under the heading " Mathematics ", showing that, in a periodical almost entirely of use only to physicists and physical chemists, it is considered desirable to include references to current investigations dealing with numerical methods which might be useful to its readers. Computing in itself is bound to be a matter of routine, but the theory on which it is based is of great interest and importance; anyone who has done a large amount of it, and there are very few applied mathematicians who have been able to avoid such work, knows that it is a good example of the interplay of theory and practice, " uniting theoretical contemplation with active application ". It has often happened that numerical results have suggested theoretical investigations, for example, why in some problems there is an obviously slow convergence in the successive approximations and in others a rapid convergence; such results have in some cases not yet been completely explained. There is a further important consideration in numerical work; however satisfactory in all respects a theoretical investigation may be as an improvement on former methods, this has to be balanced carefully against the fact that no computor will willingly give up a method with which he is familiar, whose vagaries he knows, and whose dangers and difficulties he has learnt how to surmount, to learn a new method. Even if a new method involves a smaller number of operations such as addition, multiplication, division, and the taking of square roots, this does not always compensate for the expenditure of time and energy involved in acquiring the new technique.

As a result of the increasing use of numerical methods, there has grown up a demand for more detailed and more extensive mathematical tables. In Great Britain the long-established Mathematical Tables Committee of the British Association has continued its work in collaboration with the Royal Society, and its results are now available in a series of volumes which appear at intervals as each is completed. In America similar work is being carried out by the Computation Laboratory of Harvard University and by the National Bureau of Standards respectively. The provision of such tables would have been impossible without the aid of calculating machines, which have in the last twenty or thirty years become indispensable in many university departments. It has, indeed, been suggested that in fifty years tables of ordinary logarithms will be superseded by such machines, which will be used for addition, multiplication and division. This may well be a correct estimate as regards logarithms up to 10 or 12 places of decimals, but a 5- or 6-figure table for use in the absence of a machine is likely always to be part of the necessary equipment of any scientist.

In a large group of problems in applied mathematics the solution of the equations, algebraic or differential, subject in the latter case to specified end or boundary conditions, can be replaced by the equivalent minimum theorem —the simplest case being that by which positions of stable equilibrium are determined by making the potential energy a minimum—and it is the minimum theorem which is used in obtaining a numerical solution by successive approximation. One procedure by which this solution is obtained in the case

of differential equations is known as the Rayleigh-Ritz method, and has been extensively used in the case of linear differential equations.

An important change in the syllabus of the mathematical curriculum in universities has been the introduction of the theory and use of vectors. This is due almost entirely to the enthusiasm and missionary zeal of a few applied mathematicians, in particular to E. A. Milne, whose death last summer at an early age removes from our midst one who was distinguished as much by the audacity with which he put forward new theories as by the care he took in their detailed mathematical development. It is fortunate that Milne was able to complete in his *Vectorial Mechanics*, which had been planned many years ago, a presentation of the theory of vectors which will take its place alongside those books whose authors have contributed to the understanding of the subject and have not confined their treatment to its utilitarian aspect. The theory of vectors, in its geometrical form in Euclidean space of three dimensions, is due to Gibbs and Heaviside, building on ideas and nomenclature derived from Hamilton. Its early applications were confined chiefly to applied mathematics, in particular to three-dimensional statics and dynamics, and to electro-magnetism. It is a highly specialised production, suitable only to our space of three dimensions, its laws are not those of ordinary algebra, and it has no place among the many algebras of the pure mathematician, but it does do in a concise and satisfactory manner what it sets out to accomplish, which is to provide, in Gibbs' own words, " an algebra which should be primarily geometrical and not arithmetical ". There is always a temptation to use a new tool, available for many purposes, to excess, and there seems to be a danger of this happening in the case of vectors. It takes some time for a student to learn the technique of vectors with its scalar and vector products, and the derived products known as the scalar and vector triple products. It needs practice to be able to choose from the vectors occurring in the problem under consideration those that will form products which are useful for the purpose in view or have a special significance. Bearing this in mind, I should be in no hurry, in any branch of applied mathematics, to introduce the student to vectors and their use, until the principles of the subject, and the usual methods of solution of its problems, have been learnt and thoroughly understood. In particular, I doubt if there is much to be gained by using vectors in the solution of two-dimensional problems as, to judge by examinations, their use in this connection results in little, if any, saving of time and paper. After all, a two-dimensional problem should be simpler to deal with than one in three dimensions, and it is out of place to introduce a vector perpendicular to the two-dimensional plane in order to convert the problem into a three-dimensional one, and therefore appropriate to treatment by vectors.

The proper concise notation for two-dimensional applied mathematics is the complex number $z = x + yi$, which has the advantage of conforming to the laws of ordinary algebra, and I should remind you that the real and complex numbers, in the latter case with units $(1, i)$ are the only ones that satisfy all the laws of ordinary algebra, associative, distributive, commutative, and in which division is always possible and unique. If we are allowed to drop any one of these laws, for example, the commutative law of multiplication, we have to use numbers with more than two units, and then the only possible number is Hamilton's quaternion, written in the form $u1 + xi + yj + zk$, with units $1, i, j, k$, which obey special laws of multiplication. Quaternions will do in three-dimensional Euclidean space all that vectors will do, but the test of time has clearly shown that the geometrical theory of vectors is simpler than the arithmetical theory of quaternions and, being sufficient for the purpose for which it was invented, has been preferred to quaternions.

Somewhat later in the period under review, an analytical theory of vectors was introduced and is becoming a strong competitor to the geometrical theory. This analytical theory, which deals with the components of the vectors and not with the vectors themselves, is part of the general theory of tensors and, since it deals with Euclidean space, has been given the name of cartesian tensors. It has been argued that it is better to acquire familiarity with a theory which can be generalised to apply to non-euclidean space of any number of dimensions rather than to spend time on one which is restricted in its applications. Examples of this dilemma, though not strictly analogous to the case just mentioned, confront the teacher from an early stage; there is always the question whether one should make use of the elementary notions of coordinates to start with the general definition of the trigonometrical ratios, or should work from acute through obtuse angles to the general definition; again, should one deal with determinants of the second, third and possibly fourth order only, by making each case depend on the preceding one, or consider the general determinant, with the notions of inversions and of even and odd permutations. I suppose that the answer mainly depends on the time available and one's estimate of the quality of the students concerned.

At present it is usual to develop the presentation of the analytical and the geometrical theories of vectors side by side; in Milne's *Vectorial Mechanics* the emphasis is on the geometrical theory, while in the Jeffreys' *Mathematical Methods* it is on the analytical theory. If one may venture into the realm of prophecy, it will be the analytical and therefore abstract theory that will survive rather than the geometrical and therefore visual theory, a result that would be in conformity with the trend of mathematical teaching in the last fifty years.

I fully realise that I am getting perilously near talking about mathematical notation, which Professor Neville chose as the subject of his address to Section A of the British Association last year. It is not easy to become the possessor of a copy of this address, but I would advise any of you who may come within reach of it not to pass it by in the mistaken idea that it is beyond your range of appreciation. I cannot see how some reference to notation can be avoided; the mention of vectors leads immediately to it, but nothing that I shall say will spoil your reading of Professor Neville's masterly account.

There are a number of cases where it is both convenient and profitable to use a single symbol to represent a set of symbols, as, for example, in the theory of matrices. The proof of the pudding is in the eating—if the single symbol can be swallowed whole and be easily digested, well and good: if not, it is better to divide it up into a number of symbols and so spread out the digestive process. In the case of the tensor calculus, Weyl has given his view of the matter in no uncertain terms. In his *Space, Time, Matter*, he says: "Various attempts have been made to set up a standard terminology in this branch of mathematics involving only the vectors themselves and not their components, analogous to that of vectors in vector analysis. This is highly expedient in the latter but very cumbersome for the much more complicated framework of the tensor calculus. In trying to avoid continual reference to the components we are obliged to adopt an endless profusion of names and symbols in addition to an intricate set of rules for carrying out calculations, so that the balance of advantage is considerably on the negative side. An emphatic protest must be entered against these orgies of formalism which are threatening the peace of even the technical scientist." This is the argument against the indiscriminate use of single symbols, and I could name one book to which in my view Weyl's strongly-worded criticism would apply; the decision as to whether a notation is convenient, appropriate, and useful must always be

a matter of personal opinion : should, for example, the linear vector functions of Gibbs and Hamilton, in which single symbols are used, be branded as orgies of formalism or not?

Throughout this address I have emphasised the increasing tendency in the last fifty years to replace geometrical by analytical methods in the teaching of mathematics. I must now attempt to say something about geometry itself, and here, since I am no geometer, I have not hesitated to obtain expert opinion on the changes that have taken place as regards this branch of mathematics. The geometry of the period before the reform of the Mathematical Tripos has largely disappeared except in its more elementary parts, which are represented now in the sixth form work in schools and, as regards examinations, in scholarship papers and those set in Part I of the Tripos. In its more advanced parts, geometry in the university curriculum has conformed to the general tendency to become more abstract, and hence has become less a series of tricky problems, and now requires more intuitive insight and understanding. In 1907 it was the teaching of analysis that had to be overhauled and completely recast, with the result that, owing in a large measure to the enthusiasm and pre-eminence of Hardy, the best of the younger mathematicians were attracted to analysis ; thus in a short time it became established as the chief study in pure mathematics at Cambridge and in the other universities of Great Britain.

Analysis is now about the only means of tackling the problems of applied mathematics, and its importance as a tool in other branches of pure mathematics is almost as great. It is almost certain that analysis will continue to occupy its present fundamental position ; nevertheless, there are indications that geometrical methods, using the term " geometrical " in its widest sense, are regaining their importance. Many of the problems of analysis, including Cauchy's theorem, in order to be fully understood, require an insight into topology, and topology, which is the most general kind of geometry now studied, rivals analysis in its importance in pure mathematics. That topology can play its part in applied mathematics is shown by the fact that its methods have recently been used in the treatment of electric circuits in industrial problems. In classical algebraic geometry at the post-graduate level there has been for a considerable period in this country an active school, created by H. F. Baker, which has contributed to the theory of surfaces and of loci in higher space developed by the Italian geometers. More recent developments in this field have been largely due, in equal measure, to the great advances made in both algebra and topology.

So far I have dealt with the changes in the curriculum as set out for mathematicians, in which the most important has been that of digging deeper into the foundations in order to provide a better basis for the whole structure, but it must not be forgotten that at the same time the digging had extended over a wider area. Teachers have had to broaden their outlook in order to meet the increasing demand for mathematics in most of the sciences ; they have had to cater for engineers, physicists, chemists, and biologists. A consideration of the problems to be faced in this connection would require more time than can be given to it, and I must confine myself to a few remarks. Those who talk of mathematics for physicists, for chemists and so on, are doing a disservice to their subject ; mathematics is not a function of those to whom it is being taught, and the intellectual standard should be the same for all. This does not mean that what is taught should be the same ; there is a very wide choice, and there should be no difficulty in drawing up a lecture syllabus for any type of audience which meets its utilitarian needs, and at the same time emphasises the fundamental mathematical ideas. There can, in fact, be very few of the less recondite basic theorems of algebra and analysis

which have not an immediate application to the main subject, be it engineering, physics or chemistry.

As a result of the closer relationship between mathematics and the other sciences, there has been among mathematicians a gradual realisation that no distinction should be made in the earlier stages of the curriculum between pure and applied mathematics. Specialisation in advanced work and research is inevitable, and has become more marked; just as we can no longer expect to find a Kelvin or a Rayleigh in whom both theory and experiment are represented at the highest level, so it is unlikely that there will be another Poincaré, who has been called "the last universalist", equally at home in astronomy and theoretical physics as in pure mathematics. The applied mathematician is bound to possess very much more than a mere bowing acquaintance with pure mathematics; that I take for granted, but I venture to suggest that pure mathematicians, whom some look upon as almost a race apart, have, possibly due to the demand for an expert knowledge of mathematics arising from two World Wars, attained an understanding of applied mathematics which is no less than their colleagues' understanding of pure mathematics. This is my impression of the young pure mathematicians with whom I have come in contact during the last twenty years or so. It is true that in a university it is necessary for administrative and other reasons, and to keep a proper balance, to label us as pure or as applied mathematicians, but it should be understood by the outside world that this is merely a convenient grouping and does not connote any fundamental difference. As regards essentials the two groups are united; at the same time each group has the necessary liberty in non-essentials, and there is no reason to doubt that in their dealings with each other they will continue to show charity in all things.

Here I must add that with another President you might have had an address which would have taken as its theme the fusion of pure and applied mathematics into mathematics. A few weeks ago in a conversation, in reply to my statement that the replacement of a geometrical outlook by an analytical one was the greatest change in my time, the remark was made that the gradual coalescence of pure and applied mathematics was the most important feature of the last fifty years. As things are, both you and I must make the best of it, but I have at any rate not overlooked this point of view.

Any one of the topics I have singled out for special consideration—numerical methods, vectors, geometry, the teaching of mathematics to non-mathematicians—would be a suitable subject for an address devoted entirely to it. In dealing with all of them together there is the danger that I have said too little or too much about one or more of them; but, worse than that, there are serious omissions, for example, algebra, statistics, hydrodynamics, in my list of those branches of mathematics in which there have been for one reason or another, great changes during the last half-century. The teaching of hydrodynamics has been so modified by its applications in aerodynamics that it can no longer be reproached with dealing mainly with fluids which do not exist in nature. As regards statistics, its field of operations has been extended far and wide, and a list of topics which have been brought within the scope of numerical treatment is a long and imposing one. It is, in fact, easier to enumerate some which have not so far yielded to scientific enquiry of this type; Professor M. G. Kendall instances as such Poetry, Drama and Theology, but he adds, "One never knows what may happen." Anything I could have said about those I have omitted would have been so very much at second-hand that I have followed the precept that "discretion is the better part of valour", and have not attempted to include them.

This brings me to one other matter, with which this address will be brought

to an end. We are all familiar in these days with the view that our life in all its aspects depends only on the social and economic conditions in which we exist. Thus intellectual effort arises solely from our environment, and hence it is necessary to bring all mathematics into close connection with the world outside ourselves. No one would deny that the study of mathematics has, time and time again, received strong doses of stimulants from new external phenomena, either observed or discovered by experiment, and that this is certain to happen again in the future. If as a result a new branch of mathematics should come into being, then this would be a sufficient justification for its investigation and development; but it is not necessary that the new branch should have arisen in this way, as can be shown by considering a well-known case.

Some years ago a young enthusiastic devotee of the doctrine stated above told me that non-euclidean geometry owes its origin to measurements of the angles of a triangle carried out by Gauss. It is true that Gauss was from 1821 to 1848 scientific adviser to the Hanoverian and Danish governments in connection with an extensive geodetic survey, but I cannot believe that Gauss, of all people, could ever have thought that the measurements would be sufficiently accurate to decide whether the sum of the angles of a triangle was 180° or not. Actually the difference between the sum and 180° was found to be within the limits of error of the observations. If Gauss ever did suggest that such measurements would give a definite result, I think it must have been " off the record " and not said officially. We know from his correspondence that, as early as 1799, he had worked out some of the formulae of the hyperbolic geometry, in which the sum of the angles of a triangle is less than 180°. In another letter of 1824, Gauss writes: " The assumption that the sum of the three angles of a triangle is less than 180° leads to a peculiar geometry entirely different from ours, a geometry completely self-consistent which I have developed for myself perfectly satisfactorily, so that I can solve any problem in it with the assumption that a constant is determinate, this constant not being capable of *a priori* specification." And, in the same letter: " Hence I have sometimes expressed in jest the wish that the euclidean geometry were not true, since then we should possess an absolute standard of measurement *a priori*." The letter finishes with the request: " Please regard this as a private communication of which public use, or use leading to publicity, is not to be made in any way. If at any time I acquire more leisure than in my present circumstances I shall perhaps publish my investigations." This Gauss never did, so that he had to give place to others as regards the publication of a non-euclidean geometry. You will, I am sure, agree with me that this historical excursion demolishes the idea that non-euclidean geometry arose from any connection with physical measurements; moreover, the Swiss mathematician Lambert, who died in the year 1777, in which Gauss was born, knew that the area of a triangle in hyperbolic geometry was $K^2(\pi - A - B - C)$, where K is an absolute length, and that this geometry was the same as that on a sphere of imaginary radius.

There is, however, more to be learnt from the statement that I have taken as a text on which to base the above sermon. It shows a complete ignorance of the history of non-euclidean geometry, since from the beginnings of geometry Euclid's fifth postulate, the denial of which leads to non-euclidean geometry, had been suspect. It is therefore more important to notice that the statement ignores altogether the power of the human intellect for abstract thinking and for generalisation. Pure mathematics is firmly established on these attributes of the mind, and the example of non-euclidean geometry, and there are other examples to which I have not referred, shows that in the exercise of these attributes there need not be any connection with the external

world. This dual aspect of mathematics as arising on the one hand solely from the use of the intellect and, on the other hand, from the investigation of natural phenomena, is nothing new, and has been characteristic of our science from its earliest days.

If, as is generally accepted, the origin and development of geometry was due to the practical necessity of replacing the landmarks destroyed by the periodical inundation of the valley of the Nile, this is balanced by the theory of proportion of Eudoxus, surely as abstract a presentation of mathematical theory as one could wish for, which is reproduced in Book V of Euclid's *Elements*, and is the starting point of one of the modern theories of irrational numbers. If we feel overwhelmed by what was accomplished in applied mathematics in the period 1750–1900, to which in turn French, German and British mathematicians made fundamental contributions, we must remember that in the same century and a half equally important progress was made in pure mathematics in all its branches, arithmetic, algebra, geometry and analysis, to which again mathematicians from France, Germany and Britain made outstanding contributions. If we quote Fourier to the effect that " The profound study of nature is the most fruitful source of mathematical discoveries. Not only does this study, by offering a definite goal to research, have the advantage of excluding vague questions and futile calculations, but it is also a sure means of moulding analysis itself and discovering those elements in it which it is still essential to know and which science ought to conserve. These fundamental elements are those which recur in all natural phenomena," then we must not forget Jacobi's reply, when reproached for wasting his time on elliptic functions when there were still problems in the conduction of heat to be solved : " It is true that M. Fourier has the opinion that the principal aim of mathematics is public usefulness and the explanation of natural phenomena ; but a philosopher like him should know that the sole end of science is the honour of the human mind, and that under this heading a question about numbers is worth as much as a question about the system of the world."

Thus pure mathematics has its roots in abstraction and generalisation, while applied mathematics draws its sustenance from theory and from experiment or observation, and these combine to form the main trunk of the tree of mathematics. As members of the Mathematical Association we are indeed fortunate in having chosen to devote our lives to a subject of so fundamental a character, whose history goes back to the earliest period of which we have any record, which has almost unlimited applications, and for which there is a growing demand in almost all types of human endeavour. This Annual General Meeting will fulfil its function if it provides, through its official activities and, in greater measure, by its opportunities for discussion and the exchange of opinions, the stimulus we all require from time to time in order that, as teachers, we may do our duty to our pupils and be able to tackle our ever-increasing responsibilities.* H. R. H.

* My thanks are due to Professors W. V. D. Hodge and T. A. A. Broadbent for their very kind assistance in the preparation of this address, and for their criticisms and corrections of the original draft.

WORK FROM MODERN SCHOOLS.

[At the Annual Meeting, Bristol, 1951, an exhibition of work done in Modern Schools formed a notable feature, and members of the Exhibition Committee composed a team which explained the nature and scope of the exhibition. Although the collection has now been dispersed, a careful register of its items has been made by Miss L. D. Adams, 10, The Avenue, Clifton, Bristol 8, and members who may wish to inquire about some particular item are requested to communicate with Miss Adams.]

Miss L. D. Adams : I must as convener of the exhibition committee spend a few minutes on acknowledgements of all the help we have received. First of all, of course, I must thank the Education Authority and teachers of Bristol for their support. Without the hard work, skill and, above all, the goodwill of a body of teachers in Bristol, this exhibition could hardly have been undertaken. Then the Principal, staff and students of Redland Training College have given us their support. This has been focussed by Mr. James, the mathematics lecturer, who is also the secretary of the exhibition committee. I can only say of him that he never forgets anything, and he remembers all sorts of things no one else has thought of. He has put in a vast deal of really heavy work. Many friends have helped us from outside Bristol, particularly Training College lecturers and my former colleagues, through whose agency we have secured exhibits literally from the three corners of England, Cornwall, Kent, and a little jewel from Newcastle, as well as from many places between. These add to the balance of the exhibition, for example, by introducing items from a rural environment, and they add to its interest for Bristol teachers themselves.

There are one or two general points about the exhibition which I should like to bring to your notice. This exhibition is no flash in the pan. Its origin is a small and constantly changing gift and loan collection which has been used at Teachers' Courses for many years. There is, for example, an item dated February 1936. Work of the kind shown was so well established in the old Senior School that even the war could not kill it. I want you, therefore, to distinguish clearly between the nature of the work shown and its presentation.

The work is actually work which has been going on in the schools. It has not been specially tried out this year for exhibition.

The presentation is another matter. In the showing of the work as a whole we have had the guidance, and for our notices the help, of Mr. Uncles, the Art Lecturer at Redland Training College. We are infinitely grateful to him. The individual items, on the other hand, have been most carefully selected, mounted and sometimes reproduced to make them clear for you to see. Generally the folders include the work as it was done by the children and the wall exhibits selections. This care in presentation is an expression of Modern School teachers' pride in their work and part of Bristol's welcome to the Association.

May I say again that projects have not been undertaken for your benefit. Moreover, poor work has sometimes been selected in order to show you the range of ability with which Modern School teachers have to deal.

Another general point of interest is that the teachers concerned are seldom specialist teachers of mathematics. For example, in the girls' school shown as a unit in Room A, fifteen teachers take part in the teaching of the nineteen forms. Again, one exhibitor who is in charge of mathematics in an important school told me that he had not studied the subject as a subject since taking School Certificate.

The work shown is worthy of your attention because it has been under-

taken by teachers who know how to harness the interest and energies of their pupils, and who try to use mathematics as a means of education. Regrettably they often have to undertake a great deal of remedial work. Even their best pupils are those who have " failed in the selection examination ". I believe I should say " selection test ", but the word " test " is, in my opinion, camouflage, and it is an arithmetic examination which is the bugbear of these young children. The damage is psychological, besides being damage to the treatment of the subject. Public opinion, which is felt by the children themselves, and tradition demand so much in primary schools in preparation for selection tests, and demand it of all sorts of children, that primary school teachers find it hard to break away from a mere teaching of arithmetical skills in preparation for examination and to have proper regard for their pupils' tender age and abilities. In my opinion even the α pupil is in consequence suffering from over-emphasis on teaching how and memorising, rather than on finding out and learning from experience.

There is in the exhibition in Room B a small selected junior exhibit. This is included for three reasons. In the first place, it shows work from a few schools which have made an attempt to teach mathematics as a living rather than a dead language. It consists mainly of recording and expression, related to the children's experience and imaginative efforts based on experience. This is a vital step in becoming familiar with the written symbols of mathematics. Indeed, you will find many of the Modern School exhibits show what might be called descriptive rather than manipulative mathematics. This is partly because in the Modern School new symbolism is being introduced, but also because remedial treatment has to be given to the arithmetic which has been staled in the primary school without any fullness of meaning or mastery.

To sum up this first point of the junior exhibit, there is an attempt to show one vital aspect of " practical work ", namely experience as a source of mathematical ideas, notations and simple techniques, rather than practical work as a field of applied mathematics.

The second reason for including a junior exhibit is connected with my earlier remarks. It is in the junior school that you see most clearly the distinctions in ability.* The α pupils are still with us, and their mental ages at 10+ may exceed all but those of the very ablest and oldest modern pupils. At the other end of the scale there is the C group, which includes a range of ability as wide as that of the whole grammar school population, besides a bewildering diversity of special social problems. At 10+ the mental age of a C pupil may be that of a normal infant, though in his interests he is likely to be much more mature. There are two exhibits staged in Room B, in the All-Age school unit shown there, which show this divergence of ability at 10+. In this small school C and α pupils are in the same form.

Finally, by means of the primary exhibit we can offer you two illustrations of continuity, one in the All-Age school unit, and the other from the Junior Girls' School. The work shown in the latter was collected some years ago, and the girls in the junior photographs are now in the school exhibiting as a school unit in Room A. Actually continuity in this school can be traced

* The code α, A, B, C corresponds to the quartiles of ability distribution, α being equivalent to Grammar School ability, and A, B, C approximately the codes for the 3 streams of a 3-stream Modern School. Comparing the highest and lowest quartiles of the ability distribution curve gives, with the omission of " ineducables " and sometimes of special school pupils an equivalence in diversity of ability between the C stream and the Grammar School population as defined by the code. The B stream, which is below the average of the population as a whole, contains the " median pupil " of the Modern School population and can in general be treated as the average Modern stream.

through to the infants' stage. The same might have been done for the boys' school, but the continuity for the boys is a little harder to trace because boys' schools of all kinds were so much more upset by staffing changes during the war.

Of my team, Mr. North is master in charge of the top class at the school exhibiting in Room 1 ; mathematics is not the only subject he teaches : it is not even his first interest. Mr. James I have already introduced. Miss Sowden is a member of the exhibition committee, as well as the Bristol Branch committee and the Association's Teaching committee. Her pupils' work is exhibited in several sections of the exhibition.

Mr. North : I wish to refer specially to Fourth Year work. The exhibits in Room 1 and Room A have been the work of boys and girls in one of the largest Secondary Modern Schools in Bristol. The school is situated in a Bristol Corporation Housing Estate, with a population as large as that of Taunton. The estate was originally built to accommodate people from several slum-clearance areas in the older parts of the city. The parents are of the manual working class.

In the boys' section of the school, every year is divided into five streams, from *A* to *C*-special. This gives a total of 20 classes with an average of 36 boys to each class. The time given to mathematics in the boys' school is approximately 200 minutes each week. (It is less than this in the girls' section.) We lose 20 boys each year in the second age group to technical schools, this being a second skimming, the grammar schools having taken the higher range intelligence boys in the junior school stage.

The aim of the first two years is to consolidate the work begun in the junior school The mathematics in the lower streams is of a practical nature, and wide use is made of Individual Cards. In the higher streams symbolic arithmetic and geometry are introduced. The work is within the boys' experience, and there is every encouragement for them to bring data to assist with the work begun at school. I have in mind a study made of the pre-fabricated house, carried out by the boys in a First Year Second Stream. All the information for this study was provided by the boys, many of whom lived in this type of house.

By the time the boys have reached the third and fourth years they have acquired some confidence in the use of number. There is a wider range in the scope of the work, and individual studies are carried out in the fourth year, depending on the boys' choice, need and ability. The mathematics is often integrated with social studies ; the work on view dealing with agriculture is of this type. The fourth year has been my particular responsibility for over three years. The mathematics at once presented a difficulty. Very few of the boys felt the need for mathematics in their after-school years, and I had to present it to them as a living subject which entered into every field in the outside world. Even so, the keynote of my work was usable rather than useful. The studies were rarely of a vocational nature.

The school year starts in September, and my own fourth year class is over 40 in number. Individual work is an impossibility. During this first term I teach the theory of indices and logarithms, transformation of formulae, Pythagoras and its applications, the trigonometrical ratios and the use of trigonometrical tables, and graphs of the type $d = 16t^2$. After Christmas my class is just over 30, and individual work is now begun. As far as possible I allow the boys to choose their own subjects. The time given is usually a month to six weeks, but longer is allowed if necessary. I provide as many books as I can to assist the boys in their particular assignments. In addition, they have access to the school library and local Corporation libraries. The boys sometimes work in pairs, but often alone. They come to see me each day in turn, and we talk over the work done and discuss future activity.

Now follows a list of individual work carried out during the last few months and now on view.

Pure mathematics. 1. Trigonometry, sine, cosine, tangent, with practical work indoors. 2. Logarithms and the slide rule, with construction of 100 cm. slide rule. 3. Formulae, substitution and transformation, formulae used in industry, etc. 4. Graphs, charts, conversion graphs. 5. Decimals. 6. Ratio, with special reference to the electrical industry.

Usable mathematics. 1. Air navigation. 2. Sea navigation. 3. Sun and shadows, with practical work in the summer term. 4. Surveying, the cosine rule. 5. Scales and scale drawing, with enlargement of maps. 6. Small measuring, diagonal scales, the micrometer and vernier.

Mathematics connected with social studies. 1. Local finance. 2. National finance. (Two boys are carrying out a study of local government and Parliament.) 3. Personal finance. 4. Public services, gas, water, electricity. 5. Transport, rail, road and air. 6. Coal. 7. Bristol in charts and graphs.

Geometry. 1. Geometrical patterns. 2. Machine drawing.

I have given you a brief account of the type of work carried out. There was a curious link-up between two boys on different assignments. One boy working on graphs showed that a pig could be fattened up in a few months, whereas it took years to produce a killable bullock. Another boy studying formulae discovered that the amount of eatable meat from a pig was $\frac{4}{5}W$, whereas from a bullock it was only $\frac{3}{4}W$. Somehow they got together and compared their information. Then they came to me and asked if it would not be better to produce pigs at home rather than try to buy beef from Argentina.

In several of the assignments I have allowed the boys to use formulae without understanding much about them. For example, in the study "Sea navigation", the boys were working from Hogben's *Mathematics for the Million*. They reached a stage in their studies when they wanted to calculate the direct course of a ship by great circle sailing. To understand the formula required for this a knowledge of spherical triangles was required. This was beyond their limit (and mine). But they found they could use the formula given and by substitution calculate the distance between any two places, knowing the latitude and longitude of those places. Similarly in the assignment dealing with "Sun and Shadows" we used the formula

$$\tan(\text{shadow hour}) = \sin(\text{latitude}) \times \tan(\text{hour angle})$$

for calculating the angles at the base of a Moorish sundial.

Many of the assignments required some model-making, but the models were never elaborate. If they worked I was quite satisfied. In the study "Sea navigation" we made a simple astrolabe by tying a piece of hollow brass tubing to a protractor and used a weighted cotton thread for a plumb line. The same piece of apparatus mounted on a base and legs was used as a theodolite in trigonometry and surveying. In the "Sun and Shadow" work we made a Moorish sundial and drew appropriate lines for school break times, school dismissal, etc. We also constructed a shadow pole and used a six-inch nail as an upright. The boys concerned with "Logarithms and the Slide Rule" made a ten-inch slide rule with paper, and then constructed a more ambitious 100 cm. slide rule made of wooden lathes. In the "Small Measuring" section a 40-to-the-inch thread was turned on a brass rod and a circular disc divided into 25 parts attached to a nut on the end, to explain the working of a micrometer.

In conclusion, I should like to say that I found a lively interest shown in this type of work once the boys had overcome the strangeness of working on

their own. I found, too, that it gave them a greater self-confidence, and the spirit that much was possible by research and effort was developed.

Mr. E. J. James (Redland Training College) : Miss Adams has been very kind to me in her introduction. At the risk of appearing a member of a mutual admiration society, I must add one more name, that of Miss Adams herself, to the list of those to whom tribute is due. I may have been one of a slave gang, but I am sure that every one who has had anything to do with the exhibition will agree with me that the real inspiration and drive behind the work came from Miss Adams.

My purpose is to describe the use in Secondary Modern Schools of what I term techniques—topics such as surveying, navigation, technical drawing, etc. You will notice that these occupy quite a large space in our display. For some considerable time now such subjects have been part of the mathematics syllabus of many Secondary Modern Schools. They are found in both boys' and girls' schools, although the girls' schools do not appear to use them to such an extent as do the boys' schools. I think that this is rather a pity, since girls are capable of doing very good work ; the survey of the pond which is shown in the exhibition is a first-class piece of work done by girls. Another excellent exhibit is the reproduction of the Rose Window of Westminster Abbey. I am sure there are plenty of fruitful fields for girls' schools to explore.

It is interesting to note how geometry predominates in these techniques. It is probably because they can be a means of teaching geometry that they are found in so many schools.

One obvious reason for the use of techniques is that they can present the mathematics in a practical and concrete form. No one will argue over the necessity of the Modern School teacher to present his work in a more practical manner than his colleague in the Grammar School. The Grammar School child generally has the ability to cope with a more abstract approach, and often possesses sufficient intelligence to derive interest and pleasure from intellectual achievement in academic work. In the Secondary Modern School achievement must be on a more practical plane. We can, of course, find practical approaches to mathematics without necessarily developing techniques, but a technique does serve very well for tying things together, and for providing a sense of progress and achievement. I quote the statement of a headmaster of a school which teaches surveying : " Of the various branches of mathematics the most popular in the school has always been geometry, and surveying has proved to be an excellent medium for conveying geometrical ideas and principles."

The geometry of the Secondary Modern School is a geometry of properties rather than the logical system of development of the Grammar School. Formal proof plays little part, demonstration and practical use taking its place. The formal geometry has the advantage that there is undoubtedly a sense of progress. A practical geometry of properties can be a very scrappy thing indeed. Hence the value of a technique to contain the geometry we require, to use it, and to give the necessary feeling of progress.

So we see the reasons why the Secondary Modern teacher goes to the techniques as a means of teaching some geometry. Often the technique was, and, unfortunately, still is, substituted for the geometry, but I believe that we are seeing a general and significant change in attitude towards the use and teaching of techniques. They are being taught, more and more, not for their own sakes, but as a medium for conveying mathematical principles.

I would like to emphasise the importance of this. The real purpose is not only to give the child a skill in using the particular instruments and tools of a technique, and of carrying out its special practices, but to give him an

insight into what lies behind. It is possible for a child to go through a cours say, of technical drawing, and at the end of it to have no true knowledge what he is doing. He may be well-equipped for the drawing office, whe his occupation is tracing and copying. I have met students who have spe some years in a drawing office, yet did not know elementary tangent co structions. It is not our purpose in a school to give a hundred boys a cour in drawing-office practice in case one might then earn his livelihood in drawing office. It is rather to utilise the drawing as a means of showing an using properties and constructions, which this study can very well do, and d giving training in the habit of visualisation of solid forms and their plan representations.

At the same time I do feel that we should use the conventions of a technique as far as it is possible to do so without interfering with our main purpose For example, many teachers of technical drawing are either unaware of, o else deliberately ignore, the conventions for lines, dimensioning, lay-out, an so on, which are standard British practice.

If we are to get the most value from our techniques, we shall adapt them to suit our mathematics, not change our mathematics syllabus to suit a standardised technique. In surveying, the teacher will often use methods and almost invariably will have to use instruments, which the professional surveyor will scorn. Instead of a surveyor's cross for setting out perpendiculars, a variety of methods can be used, introducing different geometrical properties, such as the 3, 4, 5 triangle, the shortest distance, the angle in a semicircle, or the bisector of the base of an isosceles or equilateral triangle. The simplest types of home-made instrument are often the most useful from our point of view, and their construction can introduce fresh points. The Jacob's staff is a good example; the construction is simple and the calibration can be an excellent exercise. Trigonometry could very well be introduced through it.

It should be realised that a technique uses mathematics rather than develops it. The teacher has to provide a development. The Infant Schools have long learned that before a child can really grasp the significance of a new conception he must be given a varied and concrete experience. The less able the child, the greater is this need. So, in the Secondary Modern school a child can be given practical experience of some mathematical relationship and will often use it before he has any true realisation of that relationship. This then is, at one and the same time, an advantage and a possible danger of the use of techniques. We can put our geometry in a practical form, so that the child gets the experience he needs, but we must also ensure that the child realises that there is some geometry there, some fundamental truth on which the practice is based. Another advantage of the practical experience is that often the need for a new skill becomes apparent. A need for simple trigonometry can arise in many ways; one has already been mentioned, another is through the determination of heights.

Techniques can show, then, mathematics applied practically, and they can bring fresh mathematics to the notice of the child. There is another way in which they can be used. In skilful hands they can present the child with a variety of situations or problems which the intelligent teacher will allow the child to tackle for himself, so that a demand is made on the child's intelligence and understanding, on his ability to apply his knowledge, and on his initiative and attack. This senior counterpart to the best type of activity of the primary school can form a most valuable part of what techniques have to offer. I believe quite firmly that a child learns much more from making a mistake and then trying to find the right way than from being shown how to do it correctly the first time.

One last point I should like to make. The Association's 1938 Report on the teaching of geometry states that franker recognition should be given to the value of intuition in geometry. The use of intuition becomes doubly important in the Secondary Modern School, but I do not think it is recognised to anything like the extent that it should be. There are many properties which a child will accept quite readily when brought to his notice, and his first real geometry begins when he sees them for himself and then realises their implications. A true use of logical geometry is completely dependent on this first stage of perception. As an example of the use of intuition, a child of 9 or 10 will readily accept the properties of similar triangles in using a 45° set-square to find heights. I am not saying that he yet realises this property that he is using, but intuition always comes before reason. As adults we often go on relying on it still for our first inspiration, and then find reasons afterwards.

Miss K. Sowden (Speedwell Secondary Modern Girls' School) : The Modern School teacher has a great deal of freedom when deciding what to teach. Unlike his or her colleague in the Grammar School, who has a goal and a well-defined path leading to it, the Modern School teacher has no General Certificate examination looming threateningly over her.

There are indeed some examinations, as Mr. North has already mentioned, but these apply only to the *A* stream, and so the work of the *A* stream must, for the first 2½ years, cover certain topics, and there cannot be much deviation from the recognised arithmetic syllabus of fractions, decimals, areas and percentages. For the remainder of the school and for the third and fourth year of the *A* stream there are endless possibilities.

There are those who say that the old senior elementary school children were able to do quite advanced arithmetic on Grammar School lines and therefore they should still do it today, the syllabus being modernised by the addition of a little algebra and geometry, such as in the first two years of the Grammar School course. And so the Modern School child is treated as a slower developing version of the Grammar School child, and the Grammar School syllabus is cut down and re-shaped to fit, more or less, the Modern School child. Now, speaking without any authority and only from my own rather limited personal experience, it seems to me that this is entirely wrong : that the Modern School child is not a slower-thinking and developing version of the Grammar School child. If the Modern School children of today were kept at school until they were 20, I think that the majority of them would still not be able to follow more than the simplest logical argument. This, of course, is a point on which other opinions are held. With the exception of about 10 per cent. of the *A* stream which I am leaving out of consideration, it seems to me that the difference is a permanent difference, obviously becoming greater as we consider the *B* stream and then the *C* stream. I intend to refer more particularly to the child of average and below average intelligence, the *B*'s and the *C*'s, who together make up two-thirds of the Modern School and about half the whole population.

The first and perhaps most obvious point of difference has already been mentioned by Mr. James. It is the Modern School child's difficulty in understanding anything which is not either immediately visible and tangible or well within its own experience. They seem to have little or no power of abstract thought, and so systems of notation, for example, present great difficulty to them, although familiarity and use will render them more acceptable. An *A* stream child may not understand a process but will memorise it (a tendency against which we have to guard very carefully), but we generally find that a poor memory, the second point of difference, accompanies a low intelligence quotient, and so the *B* and *C* stream children have an additional

difficulty in having to work things out again for themselves instead of memorising them. This, of course, is a good thing in itself, but slows up excessively any work done, and when we consider that *B* and *C* stream children have very little reasoning power (the third point of difference) with which to work things out, it is not surprising that we seem at times to make such slow progress.

It is very interesting to notice the age at which ability to reason develops. I have found that most *A* stream children have it when they come to us at 11, but that most *B* stream and all *C* stream children lack it almost entirely at this age, but that it does develop very noticeably about 13 or 14; and, at least in my experience, teaching a *B* or *C* stream class becomes very much easier at this age. The children will tackle with ease and pleasure problems which a year before seemed insoluble to them. Their rapidly-increasing self-confidence, their developing personalities, their experiences outside school and in other school subjects seem to make them approach mathematics from a different angle—a practical approach, saying, "What use will this be to me?" and if the answer is favourable they will develop hitherto undreamed-of abilities.

I have in mind one particular *C*+ class, a very poor *C*+. When they came to us from their junior schools half of them could not even do subtraction and short division, and if asked to bring 6 feet to yards, they knew that the problem had something to do with 3, but whether to multiply or divide was quite beyond them to think out. For 2½ years I almost despaired of their ever learning any arithmetic, but in their third year, quite suddenly everything seemed to fall into place in their minds and they were able to work processes out for themselves, and now in their fourth year they are doing some quite difficult work for a *C* stream on feeding a family, involving working out in great detail quantities and costs such as the average amount of milk in a cup of tea and the cost of a teaspoonful of sugar. Now I am sure that the reason why so many children are classed as backward in arithmetic is that the subject is so abstract to them as to be meaningless.

The great disadvantage of a Modern School teacher is that she has never been to a Modern School as a pupil. She was taught by Grammar School methods, but I do not feel that these are suitable for a Modern School child, and an attempt has to be made to get inside the mind of the child and to find out what it really needs and how it can most readily be interested in a new subject, so that it will make the necessary effort to master it.

The recent tendency has been to put mathematics into projects, and this is obviously a very good way of stimulating interest. I do not wish to decry the project method, but I feel very strongly that the important thing is the method of approach to a subject on its first introduction. It must be concrete, practical and usable, either in a utilitarian sense or for pleasure and interest. In Room 3 of the exhibition you will find many examples of this practical method of approach. There is the experimental work of a rural school on the growing of potatoes, with results expressed graphically. There are graphs giving the results of scientific experiments. There is an exhibit headed "Social Studies", which is work done by the pupils of a geography specialist who has used mathematics to compare, contrast and emphasise the facts collected in the course of the work. There is work connected with the teaching of French, where arithmetic is done in French money and measures. There is a lot of work on Domestic Subjects: buying a house, furnishing a house, decorating a house, feeding a family, clothing a family, and a little work on rug-making. There is even a project on the lighting of a model theatre. All the work in this room expresses what I have been saying about the practical approach.

SCHOLARSHIP SYLLABUSES IN THE GENERAL CERTIFICATE OF EDUCATION AND IN UNIVERSITY EXAMINATIONS.*

Mr. P. C. Unwin (Clifton College) : As a rough division, from the school point of view, of the very wide field to be covered, Mr. Parr will deal mainly with pure mathematics, and Mr. Tongue with applied, while I, in opening the discussion, should like to remind you of a few general considerations.

The first point I would make—an obvious one, but easily overlooked in the heat of battle on detail—is the inevitability of compromise, since both types of examination serve more than one purpose. The General Certificate of Education is primarily a leaving certificate, marking the *end* of a school course, and many who take it will not go to a university. The mathematical course itself is only part of a wider scheme of general education, which should affect its content and also limit the time we can afford to give to it. (I imagine most of us subscribe to the " one-third, two-thirds " division, though I was rather shaken by the figures given at an earlier discussion by Dr. Maxwell for the time spent on mathematics alone by scholarship candidates he had questioned.)

The Certificate examination is also being used more and more for the secondary purpose of awarding scholarships, by the State and by local authorities (many of whom are now to demand evidence of ability at a higher level than before). The range of General Certificate of Education scholarship syllabuses varies enormously—from *S* papers consisting solely of harder questions on the *A* syllabus to those set almost entirely on a much more advanced syllabus. Bearing in mind the double function of the General Certificate of Education, I think the best solution is the sort of compromise adopted by the Cambridge Syndicate—a wider scholarship syllabus, but with not more than a half of each paper based on the additional topics.

University scholarship examinations, too, have a function additional to the primary one, though the emphasis is now on looking forward, to the *beginning* of a new course, and again some compromise is indicated. As an illustration of a pull in opposite directions, I recently received within a few days of each other two independent criticisms on the same point of detail in the Cambridge papers. One said there ought to be more homogeneous coordinates, because they are so important afterwards ; the other said there ought to be less on homogeneous coordinates, because they are so easy at this stage and it is very difficult to set fresh questions on them!

Mathematical education should be a continuous function right through school and university, but continuity at the join is not always obvious. Forsyth's picture of the extraordinary divergence between the changefulness of a university course and the almost sphinx-like changelessness of a school course, as indicated by papers set to schools, is exaggerated, but there is something in it. Changes there have been, but they have tended to come slowly and uncertainly, particularly where there has been no published syllabus, as with Cambridge scholarship and the Oxford and Cambridge Joint Board Distinction papers.

For one thing, it is easy to *drop* topics, though this may involve a disconcerting and unnecessary time-lag before the schools tumble to the fact that the dropping is by design and not by accident or caprice. But, in their anxiety to be fair, examiners will be much more reluctant to introduce *new* topics without warning. Warning, however, is difficult without a published syllabus, which is one of the arguments for having such a thing, not neces-

* A discussion at the Annual Meeting of the Mathematical Association, Bristol, 31st March, 1951.

sarily in great detail, but enough to serve as a guide to teachers, without unduly tying down examiners, and to allow of desirable changes being made without fuss or inconvenience.

Not the least valuable feature would be the possibility of showing topics *not* required, as in the Oxford University scholarship syllabus : " Knowledge will not be presumed of continued fractions, etc., etc. " ; or in the Cambridge General Certificate of Education syllabus : " Questions will not be set which require a detailed knowledge of the general equation of the second degree." How long was it, I wonder, before the average teacher felt tolerably certain that continued fractions had gone for good from the Cambridge papers, or, more recently, that much of the old analytical geometry (for example, the detailed work on the general conic in point and line coordinates and the metrical part of homogeneous coordinates) or the harder complex function questions had been dropped from the Joint Board papers? For the last year or two the same papers have contained practically no questions on pure projective geometry. Is this deliberate or not?

This is not a question of cramming, or of sticking too slavishly to an examination syllabus, but of efficiency of organisation and training. It is often impossible to cover the whole ground, and the teacher wants to make the best use (in the best sense of that phrase) of the time available. If a change is intended, for example, to cut out what is regarded as dead wood, or to conform to a new trend in the university course, then the teacher should know about it, in the interests of everybody concerned.

While on the subject of covering the ground, may I digress to put in a plea for the smaller schools, whose specialist work is often done under very great difficulties. I am not sure that they get enough consideration in discussions on ideal syllabuses, etc., and I should certainly like to see them represented on any committee dealing with such matters.

To return to the somewhat unsatisfactory nature of the continuity between school and university, I think the main remedy is more co-operation and consultation. That is why I think one of the most encouraging developments for many years has been the setting up of the Cambridge Joint Advisory Committee. This (whatever you think of particular points in its Report) has brought together representatives from the schools and from different faculties of the university, and has made a brave attempt to rationalise this difficult business of Sixth Form mathematics. I should like to see a supplementary report extending the syllabus beyond the present two-year course, and I should then hope that the invitation given in 1945 to the Cambridge colleges and to the Joint Board to subscribe to the agreed recommendation would be accepted.

Planning Sixth Form work would in any case be easier if there were more coordination between the groups of colleges. To give just two small examples, it seems a pity that one group should regularly set questions on differential equations and another not, that one should devote 4 or 5 questions out of 30 in the elementary paper to geometry and another 10 out of 30. After all, candidates for different groups are taken together at school, and will be doing much the same course at the university.

But these are unimportant points compared with the main issue of closer co-operation between school and university. This might be achieved through a standing committee, which would discuss common problems and keep each side in touch with conditions and developments in the field of the other. It would be ready to supply information and make suggestions and recommendations from time to time, possibly through the *Gazette* or an occasional pamphlet, for example, on methods and lines of approach, on priorities at the scholarship and post-scholarship stages, on the mathematical equipment

expected of the scientist or engineer, on trends and changes at the university and their implications for the schools in the dropping of old topics and the introduction of new ones, in shifts of emphasis, and so on.

Two changes at Cambridge since my time might be mentioned. One is the striking difference in geometry (and here I cannot help shedding a tear for the old " analytical "), which has to some extent already affected the syllabuses of examinations taken by schools. Some account of this transformation would have been very helpful earlier, and would not be out of place now. The other is the increasing tendency to take the Preliminary Examination instead of Part I at the end of the first " year ". This in practice means the end of only six or seven months' teaching, which, moreover, may follow the long interruption in mathematical work due to national service, and which will in any case include the time required to settle down in a new environment. By all means expose the bright young freshman to the stimulus and discipline of really new work. But the old plan of letting him plunge at once into second year work, and yet not making him take a stiff hurdle so soon, gave most of the benefits of the new system without its dangers. It had the further advantage of complete flexibility, the proportion of time given to second year work and Part I work respectively being adjustable to suit individual capacity. The new system involves real strain for some, and (to come nearer home) pressure arising from it may sooner or later increase the threat of over-specialisation against which we have to be constantly on our guard. This affects *syllabuses*, some aspects of the background to which I have been trying to consider.

May I add two things as a postscript. First, I missed the announcement in the *Gazette* for February 1950 ; and I have only just heard, after putting together the foregoing remarks, of the setting up of a sub-committee to discuss the relation between Sixth Form and university first-year work, one of the things I was most concerned about. This is a partial answer, but the committee I had in mind would be a cross between the Cambridge Joint Committee and the new sub-committee (which apparently includes no representatives of men's colleges at Oxford or Cambridge), and it would be a *permanent* liaison body. It would also have on it at least one representative of the hard-pressed smaller schools.

Secondly, after talking about what I feel to be a lack of co-operation in some cases, may I refer to our good fortune here in Bristol. It is only fair to say that we are a small and compact body, and so co-operation is easier. But it might not have happened, and the fact that it exists in such free and cordial measure, though I would not pretend that it is perfect, of course, says much for both sides. And with this tribute may I couple the name of my friend, your President, with whom it has been my privilege and pleasure to work, and to whom, over many years, so much of the credit for the happy family feeling between mathematicians of schools and university in Bristol is due.

Mr. H. E. Parr (Whitgift School) : I have been asked to say a few words about Scholarship syllabuses in the General Certificate of Education and University examinations from the standpoint of the schools, with particular reference to pure mathematics, both for the mathematical specialist and for the scientist.

First, let me take the Scholarship paper of the Advanced level, that is, the type of paper which is usually taken in conjunction with physics and chemistry. We have yet to see the first General Certificate of Education question papers, but the published syllabuses of the different examining bodies show such big variations that it is difficult to compare one with another. London and the Northern Joint Board retain separate Pure Mathematics

and Applied Mathematics, as well as the single subject Pure and Applied Mathematics, while Cambridge has only the single subject Mathematics. I should like to draw attention to the fact that even in the scholarship syllabus the subject Pure Mathematics at London makes no mention of differential equations at all. There is nothing about Maclaurin's theorem, or even about the *use* of the expansions for $\sin x$ and $\cos x$. It would be interesting to hear what members think. My own view is that it would be much better to have a wider syllabus, and to cut out from the Scholarship paper the sort of question which demands a high degree of manipulative skill in topics already covered by the first two papers. At Cambridge the syllabus is wider, and an opportunity is also offered to do some statistics. Now I do not know much about statistics myself; in fact, perhaps like a few other people present, I find myself constantly re-learning statistics, only to forget it again immediately afterwards because I have no one to teach it to. But I do feel that it is time statistics was introduced into the Sixth Form course, and this is a subject you might wish to say something about later. Although, as I have said, the Cambridge syllabus is wider than the London one, it remains true that there is a very large gap between it and the pure mathematics papers set for open scholarships in science at Cambridge and Oxford.

Coming to Further Mathematics, which is for the mathematical specialist, we have to recognise the fact that at such an important examining body as London there does not now exist anything comparable to the old Group III. I think the total number taking Group III mathematics at London in its last year (1949) was about thirty. Is there any demand from schools who take the London General Certificate of Education for this Further Mathematics? In the Further Mathematics scholarship syllabus at Cambridge I, personally, should find little to criticise. The most noticeable changes occur in geometry; reciprocation, for instance, has gone, and so have homogeneous coordinates.

In connection with University Scholarship examinations, it has to be borne in mind that the extension of military service from eighteen months to two years is having the effect of making boys leave school earlier than they did, in fact during their third year in the Sixth Form. It seems to me more than ever important to keep the Open Scholarship syllabuses in line with the new General Certificate of Education Further Mathematics at the scholarship level. At present the gap is more than can be filled in one term. It is no answer to say that the syllabus a boy or girl can cover in seven terms should not be measured by General Certificate of Education standards if he or she hopes to be successful at Oxford or Cambridge. County Major awards and State Scholarships are awarded on the General Certificate of Education. The promising mathematical specialist has to be taught in the same form as boys or girls who have the limited objective of General Certificate of Education Further Mathematics, and for the first two years, therefore, our work in the Sixth must be geared to that syllabus. It is the responsibility of the Open Scholarship examinations to frame a syllabus which can be covered in one more term.

In the open scholarship examinations at Cambridge there is one paper which does not really come under the heading of our subject this morning; indeed, it is entitled at one group of colleges "supplementary non-mathematical paper for candidates in mathematics". But if we at this Association do not discuss it, I do not know who else will ever do so. The paper last December began with a précis, an excellent idea, though I doubted whether it was fair to expect a boy to condense a passage of 560 words from Bertrand Russell's *Introduction to Mathematical Philosophy* into not more than 100 words. Next came a translation, a choice of Latin, French, German or

Spanish. But it is the last part of this three-hour paper to which I should like to draw your attention. Four questions out of fourteen are to be answered in short essay form, and the only ones of particular mathematical interest are :

" In what ways are diagrams likely to be both a help and a hindrance in doing mathematics? "

" With regard to *one* of the following games, indicate some occurrences and features (*a*) that could, (*b*) that could not, be mathematically interpreted and understood : soccer, rugger, cricket, lawn tennis."

" What do you mean by *either* a point of space *or* a moment of time? "

This seems to me like an opportunity missed. How much it would stimulate the mathematical specialist in our Sixth Forms if we could have some questions on the historical and philosophical aspects of the subject. For example :

" Write a short account of the chief contributions to mathematics of the Alexandrian school " ; or

" Hindu and Arabian contributions to mathematics " ; or

" The development of the calculus " ; or

" Babylonian numbers."

Professor Heilbron yesterday reminded us how fascinating is the history of mathematics, and it is not difficult to interest boys and girls at the Sixth Form level. To do so is all the more important now that the old Higher Certificate subsidiaries, especially English, have disappeared. We must encourage the specialist in the Sixth Form to find his general culture by exploring the fringes of his special subject ; by finding its points of contact with other subjects ; by tracing the actions and reactions between mathematics and the social development of mankind.

In conclusion, a word or two about the Oxford and Cambridge college entrance examinations. In these days of county major scholarships, the main thing for a boy is to get a place in a college, and in the procedure for doing this the widest diversity exists. In this respect the women's colleges, at both universities, have a much more sensible system ; but no two men's colleges are alike in the matter. Why cannot the dates be announced? Is the examination competitive or qualifying? What is the syllabus? Would the college prefer the boy who is unlikely to get an open scholarship to take the scholarship examination in order to obtain a place? Should a boy enter for several colleges at the same time? These are all questions to which the schoolmaster would like to know the answers.

Mr. F. J. Tongue (Kingswood School) : I have been asked to comment on the question of applied Mathematics syllabuses in both the General Certificate of Education and University Scholarship examinations. I shall do this from the point of view of a school teacher, and one who like the majority of school teachers in Sixth Forms nowadays has to deal not only with a small minority of mathematical specialists, but in most of his periods has also to teach scientists, engineers and people to whom the very idea of taking scholarship papers would seem fantastic. I therefore find that in practice my candidates for Scholarship papers have to do a good deal of the requisite reading on their own, with little guidance from me except in the form of notes. This circumstance, which I am sure is not peculiar to me, has a distinct bearing on the topics under discussion. It means, in fact, that one is definitely biassed against any increase in the content of the syllabuses for either the General Certificate of Education or the University examinations, and, further, that examination papers which contain too many questions on the advanced parts of the syllabus unwittingly penalise the smaller schools where classes are far from homogeneous. Therefore any increase in syllabus ought to be balanced either by an equivalent pruning elsewhere or by a greater choice of questions.

Having said all this, I ought to add that, nevertheless, I am sure that the Association will welcome any advice from teachers in the Universities as to how to modify syllabuses in accordance with present developments in University teaching.

Let me first consider Scholarship papers for mathematical specialists in the General Certificate of Education. These papers are used by outside bodies for two main purposes—the award of State Scholarships by the Ministry, and of senior County Scholarships by Local Education Authorities. Naturally the papers must provide a fairly severe test for the first purpose. My own view is that Local Education Authorities who expect measurable results from these papers are likely to be disappointed, and will scale down their requirements accordingly. It therefore seems desirable that the syllabus for such papers should point towards Open Scholarship work at the Universities.

I therefore advocate having certain topics included in the Scholarship syllabus and excluded from the Advanced syllabus. The following are suggested topics : * harder friction (for example, toppling or sliding, friction at more than one contact) ; the parabola of safety ; * shearing stress and bending moment ; the equilibrium of flexible chains ; * linear motion under varying forces (other than simple harmonic motion) ; * simple problems on the motion of a rigid body in a plane ; small oscillations (one degree of freedom). I would add that (as is now emphasised in the Cambridge Local Examination) not more than half the questions in the Scholarship paper should be on such topics ; the remainder should consist of harder questions on the Advanced syllabus. I also feel that there should be one whole Scholarship paper on applied mathematics, and at least one, and probably two or three, on pure mathematics. I would at this point put in a plea for more questions, both in the General Certificate of Education and in University examinations, on the theoretical development of the subject, for example, bookwork, and discussion of the main principles, such as Newton's Laws of Motion. Mechanics papers can so easily become a collection of problems testing not much more than the candidates' knowledge of the tricks of the trade.

I turn now to the Scholarship examinations for mathematical specialists at Oxford and Cambridge. At Cambridge there is for better or worse no detailed syllabus issued by the groups, who go their own several ways. I do hope that those concerned realise how tedious the situation is for us at the schools, when we have to prepare pupils for entry in more than one group. We should be much happier if the Cambridge groups could combine to issue a detailed syllabus, indicating more especially what topics are omitted from the examination. The syllabus issued by the Oxford Colleges does precisely this. I am sorry if my remarks exclude reference to the Scholarship examinations of other Universities. The reason is that I am not familiar with the papers or the syllabuses.

I feel that the present applied papers at Cambridge afford an excellent test ; the number of principles on which the questions are based is sufficiently restricted for the element of luck to be less prominent than in other papers, and on the whole a good balance is maintained between theory questions and problems. At Oxford I note there is a Physics paper for mathematicians ; I do not know whether it is compulsory nor what part it plays in the award. I am certain, however, that it is right that all would-be mathematicians should do physics right through their Sixth Form course, and should therefore have to offer physics in the scholarship examination. The present arrangement of an optional paper in physics, which we are told, in two of the Cambridge groups, only helps towards an award if scholarship level is obtained on it, has the consequence that many candidates for mathematical scholarships must seriously neglect their physics and may, indeed, be encouraged to drop

the subject. The more conscientious candidates are thus handicapped by this arrangement. It would be very helpful if we could know how the new arrangement of papers in the King's group is working.

I doubt whether, with the recent increase in the scope of analysis and algebra in the scholarship examination, applied mathematics should any longer occupy half of the general mathematical paper at Cambridge. I note that at Oxford there are three pure papers and one applied paper, a better proportion I feel, and one which reduces the element of luck in the pure papers. So far as the syllabus is concerned, I should add the following topics to those included in my earlier list of General Certificate of Education papers: virtual work and stability; problems on variable mass and the motion of chains; radial and transverse accelerations; motion on a smooth curve.

I now come to the consideration of Scholarship papers for non-specialist pupils. In the General Certificate of Education there is usually not more than one such mathematics paper. I am sure that there should be a very careful scrutiny of purely Scholarship topics on this paper, and that at least half the paper should be of harder questions on the Advanced syllabus. My own experience with "Further Mathematics" in the Oxford and Cambridge Joint Board Examination has been that the snippets of new work chosen do not form a coherent whole, and in many cases do not yield a good range of sufficiently easy questions. For example, in the applied part of the paper the topics of stability and radial and transverse accelerations are not amenable to very elementary treatment. Now that the paper has ceased to be an optional one and has become compulsory for many candidates, I think that its content must be carefully reconsidered. I would include *only* those topics starred in the list of Scholarship topics for specialists which I gave a few minutes ago. In the Cambridge Scholarship examinations for scientists I consider the applied paper admirable. At Oxford I think the subsidiary mathematics for science candidates is often a very uneven paper, and usually the applied mathematics on it is quite trivial.

There are a number of other general topics relating to applied mathematics that I have not had time to deal with; for example, the place of statistics and of hydrostatics. I leave these for you to discuss, and I very much hope that we may hear the University point of view from representatives of as many different Universities as possible.

Dr. W. L. Ferrar (Hertford College, Oxford): I must be brief, for I note that I have exactly *minus* five minutes in which to speak to you and remain within my allotted time. What one needs to do for boys and girls at the scholarship stage is to give them a chance to try out their power. This is the stage when, having done the elementary drill and acquired a little technique and skill, they begin to work with both profit and pleasure; and before I go on, I want to say this: "If scholarship mathematics cannot be done with pleasure, it ought not to be done at all."

The earlier stages are necessarily governed by very precise rules and syllabuses; this bit of algebra and no more; this and this function you must learn how to integrate, and no one will be allowed to ask you how to integrate anything else; these are the propositions in geometry which candidates are expected to know; and so on. This is right and proper when your task is to teach and examine a prescribed limited knowledge appropriate to the attainments of large numbers of candidates at the ordinary level. It is still well enough at the advanced level; the task of agreeing on and adequately defining in detail the subject-matter of an examination is distinctly harder at the advanced than at the ordinary level, but it can be done. When it comes to scholarship—and I expect many of you will want to rise up and smite me—I believe that a detailed syllabus is undesirable, and I am altogether opposed

to the idea of having one. If you pile on top of the advanced syllabus, this and this bit of algebra, each precisely defined; this, that, and the other item of integration; this type and that type of problem in mechanics; these particular techniques in geometry, trigonometry, algebra, calculus, and mechanics—always assuming that you can get a representative body of examiners and schoolmasters to agree on which bits go in and which bits stay out—then, indeed, you would have a body of work which boys and girls could study at scholarship level, but you would have destroyed the whole spirit and tone of scholarship mathematics: and I hope you will never try to do it.

The lack of a detailed syllabus does not mean chaos, provided that the actual persons responsible for setting questions are sensible. Gradual change in the type and character, even in the subject-matter, of a scholarship paper is desirable: sudden large changes are positively criminal, and examining bodies ought to guard against them. My own personal view is that each of the eight public examination boards should at intervals, say every five or six years, hold consultations between its examiners and representative teachers from schools that take the board's examinations.

As for the University scholarship, there is often a real difference of type between its questions and those set by the eight examination boards: but then the purposes served are different, though the recent tendency to use the scholarship examination for the admittance of commoners with local awards, or the prospect of them, makes the differences less marked than they used to be. As university examiners looking for suitable recipients of college awards, we are not really interested in your worthy, teachable, and industrious boy whom you can train to answer what he has seen before by the sheer merit of your teaching. We are interested in boys who show signs of ability, whether they have been taught by the best schoolmasters at schools with a well-organised mathematical sixth or left to work by themselves as non-conformers to the general pattern of the science sixth at a small secondary school. It is not often one sees great ability, though when it comes it is unmistakable: what one usually looks for—and praises Heaven if one sees it—is some sign of real appreciation and some sign of native wit. But one could expatiate for hours on the hopes, fears, successes and failures of university examiners. Let me, in hasty conclusion, say only this in defence of our present system: in twenty-five years I have not had more than one occasion really to regret any election I have made to a mathematical scholarship or exhibition. The elections were not all firsts—there are not that number of first class fish in the sea—but they were all men capable of reading the honour school of mathematics with profit and pleasure; and if I had to choose between the two, I should choose pleasure derived from his work as my criterion for justifying an election.

Dr. I. W. Busbridge (St. Hugh's College, Oxford) as an examiner, wishing to express the woman's point of view, pointed out the differences due to the later specialisation for girls. Very few would come from each Grammar School—one girl in five years was typical—and staffing always was a source of difficulty. This accounted for the limited syllabus in the women's colleges of Oxford. Although the physicists took some of the mathematics papers, she deplored a compulsory physics paper for mathematicians, noting that many a girls' school had no teacher of physics.

Mr. K. B. Swaine (Yeovil) distinguished between an examination syllabus and a teaching syllabus, and contrasted the restrictive nature of the former with the flexibility of the latter. Topics should be in an examination syllabus only if they were in a large number of teaching syllabuses for that examination.

Mr. Howell wished to speak for the smaller schools having only one Sixth

Form course containing, consequently, mainly scientific students. He suggested a widening of the syllabus, together with a greater choice of questions. The dropping of differential equations and hydrostatics would be a pity, since these were needed for scientists. The applied mathematics contained too much mechanics, whereas electrical theory might be justifiably included, together with some mathematics applicable to biology.

Professor Heilbronn (University of Bristol) suggested that the purpose of scholarship examinations is to test the candidate, not the schoolmaster; the University must guard against the student who is just extremely well-trained. He stated that at Bristol there was a very restricted unpublished syllabus, and as compensation the questions were hard. Referring to a point made by the previous speaker, Professor Heilbronn emphasised that differential equations were not needed before the university course. Students did need simple trigonometry, algebra and the binomial theorem.

Mr. R. Beetham (King Henry VIII School, Coventry) wished to support Mr. Parr on the need for a syllabus for the Oxford and Cambridge open scholarship examinations, a syllabus of topics to remove the strong element of luck.

Dr. H. Martyn Cundy (Sherborne School) made the point that a boarding school has too many other activities. The true mathematician was very rare, but Dr. Cundy was interested in the "syllabus for scientists". After remarking that a list of topics could replace a syllabus, he queried the desirability of including differential equations, reciprocation, binomial theory and projective geometry.

Mr. R. V. H. Roseveare (Winchester), commenting on Professor Heilbronn's "well-trained" candidate, suggested that boy and teacher might present themselves together! The good candidate could, in fact, always get through, whatever form the paper took. What was important was that easy questions on advanced topics should not be set. Questions at scholarship level should be hard—let the examiners set what they liked with a choice of, say, forty questions.

Mr. B. H. Chirgwin (Queen Mary College) thought it desirable to have a syllabus for engineers and scientists and to have applied mathematics for non-mathematicians. He agreed with Professor Heilbronn that a thorough grounding was most important. As an instance of desirable widening of applied mathematics he suggested the substitution of statistical concepts for, say, "harder friction".

Dr. E. A. Maxwell (Queens' College, Cambridge) said that the fundamental difficulty was that the universities were laying emphasis on picking out the exceptional man, whereas the schools had to train him with the other pupils. Co-operation was needed. The teachers' duty was to avoid undue cramming, while the universities should keep in touch with the point of view of the schoolmaster. Dr. Maxwell did not approve of easy questions on advanced topics outside the scope of the student's age, and although outlandish topics might be fair, misleading emphasis should not be laid on any such topic. He admitted responsibility for the paper mentioned by Mr. Parr, and referred to the supplementary papers, which were designed to test a student's general range of ability and to find if he had been crammed in mathematics only.

Mr. Robbins (Sedbergh) favoured a list of topics, but thought that universities might indicate, for example, whether vector analysis was in the first or second year of their course. He thought some work on matrices should be done at school.

A member asked whether statistics still formed part of the syllabus for Cambridge and for other universities. Replies that it did were given for the Northern, Bristol and London Universities.

Replying to the discussion, **Mr. Unwin** said that he did not expect all bodies to have the same syllabus, nor that each syllabus should be issued in great detail. He did want to know the range of each syllabus and information of changes. He asked if Dr. Ferrar objected to specific topics. **Mr. Parr** was not aware that Dr. Maxwell had set the paper to which he had referred. A non-mathematical paper might well be set to test for " decent English ", but the candidate should be given interesting topics. **Mr. Tongue,** referring to the supplementary papers in physics, considered a boy ought to be tested on what he had been doing, whether it was physics or classics. He denied that any boys were having 25 hours allocated to mathematics and 8 hours to other subjects, and thought the proportion was more like 16 hours and 15 hours. **Dr. Ferrar** agreed that a list of topics might be helpful, but thought that the existing papers always picked the Exhibitioners aright, and with that he was satisfied.

GLEANINGS FAR AND NEAR.

1676. The application of the calculus of probabilities to the exact sciences also involves many difficulties. Why are the decimals of a table of logarithms or of the number π distributed in accordance with the laws of chance? I have elsewhere studied the question in regard to logarithms, and there it is easy. It is clear that a small difference in the argument will give a small difference in the logarithm, but a great difference in the sixth decimal of the logarithm. ... But as regards the number π the question presents more difficulties, and for the moment I have no satisfactory explanation to give. —Henri Poincaré, *Science and Method* (English translation, Nelson and Sons), p. 88. [Per Mr. H. W. Chapman.]

1677. To one whose admiration of the British engineer had long been based on the good sense with which he adjusted his result to the right answer by five applications of the two times table, it came as a shock to find that the foreigner had only to make as many applications of the one times table :

$$981 = 11 \times 111.$$

[Per Mr. C. B. Gordon.]

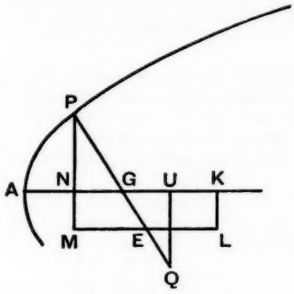
1678. Dr. George North, the Registrar-General, explaining yesterday the purpose of the Census—the first for 20 years—said : " It is wrong to say that we are expecting 99 per cent. of the population to complete the forms. That would leave a horrible hole. We expect to get a 100 per cent. picture or at least a 99·9 recurring.—*Daily Express*, March 6, 1951. [Per Miss H. Bromby.]

1679. Here in Switzerland postage surcharged is based on the factor 20. Thus, a letter stamped with 2½d. instead of 4d. is surcharged at the amount of the deficiency, 1½d. multiplied by 20. This comes to 30 centimes, or 6d. —Letter in *Daily Telegraph*, December 28, 1950. [Per Mr. F. M. Goldner.]

1680. When children travel with adults, three children may occupy a double seat only if the number of adults in the party does not exceed six, otherwise each child counts as an adult and must have a seat to himself. This means that I can take a party of 39 adults and 39 children on two 33-seater coaches, provided that one of them is occupied by adults only, but if three children on one change places with two adults on the other then both are overcrowded and twelve children must be left behind.—Letter in *Manchester Guardian*, January, 1951. [Per Mr. A. P. Rollett.]

MATHEMATICAL NOTES.

2216. *Elementary proof that the feet of the normals from a given point to a given parabola are concyclic with the vertex of the parabola.*



In the diagram, P is the foot of a normal from Q; QU is an ordinate, $UK = NG$, $KL = \frac{1}{2}UQ$, and $KLMN$ is a rectangle. It follows that L is the same for the three positions of P; and also that $ME = EL$.

$$PN : AN = 2NG : PN = 2ME : PM = ML : PM$$

and so APL is a right angle, and the circle on AL as diameter passes through P.

This may be compared with the analytical proof which combines the condition

$$y : 2a = y - g : f - x$$

for the normal at (x, y) to pass through (f, g) with the equation

$$y : 2a = 2x : y$$

of the parabola, giving

$$2x : y = 2y - g : 2a + f - x$$

i.e.

$$x(x - f - 2a) + y(y - \tfrac{1}{2}g) = 0.$$

A. R.

2217. *Repeated factors in the linear equation.*

1. $$(D - a)^2 e^{mx} = (m - a)^2 e^{mx}$$

Differentiate w.r.t. m

$$(D - a)^2 x e^{mx} = 2(m - a)e^{mx} + (m - a)^2 x e^{mx}.$$

Put $m = a$

$$(D - a)^2 x e^{ax} = 0.$$

2. $$(D^2 + n^2)^2 \cos(px + q) = (p^2 - n^2)\cos(px + q).$$

Differentiation w.r.t. p gives a new solution of $(D^2 + n^2)^2 y = 0$. This method avoids the use of imaginary quantities or the "assumption" of special forms.

3. A triple factor can be treated by two differentiations, etc.

4. When solving the Euler Homogeneous equation by assuming $y = x^n$, the case of repeated roots in the equation for n can be treated in a similar manner.

5. The argument of 1 and 2 above can be used to introduce students to the case of equal roots of the indicial equation in the Frobenius method of solution in series. (Piaggio *Differential Equations*, 97.)

H. M. FINUCAN.

2218. *The inverse square law.*

The treatment of rectilinear motion under the inverse square law of acceleration is along the following lines in most textbooks :

$$\ddot{x} = -k/x^2, \quad \ldots\ldots(1)$$

hence

$$\dot{x}^2 = 2k\left(\frac{1}{x} - \frac{1}{c}\right). \quad \ldots\ldots(2)$$

If c is finite and positive (2) gives no real value of $\dot{x}$ if $x > c$; this agrees with " common-sense " and with energy considerations. Unfortunately (2) also fails to give real values of $\dot{x}$ when x is negative ; this does not agree with " common-sense " and energy considerations.

The fact is that (1) is correct only for positive values of x ; it should be replaced by :

$$\ddot{x} = -k|x|/x^3 \quad \ldots\ldots(1^*)$$

whence we deduce *for all values of* x

$$\dot{x}^2 = 2k\left(\frac{1}{|x|} - \frac{1}{c}\right). \quad \ldots\ldots(2^*)$$

The moving point oscillates between $x = +c$ and $x = -c$; the space time curve has the form of a cycloid with alternate arches inverted.

H. M. FINUCAN.

2219. *On Note* 1202. *Special relativity and the Lorentz transformation.*

It has become a commonplace of mathematical physics that the Lorentz transformation can be represented as a rotation in a Complex Euclidean plane defined by the space-coordinate x and the time-coordinate $\tau = ict$ where $i = \sqrt{(-1)}$. C. T. Rajagopal and I showed in the *Gazette* (Vol. XX, No. 239, pp. 208–9) that the metric gauge-conic (Cayley's " absolute "), left fixed by the Euclidean transformation

$$x' = x\cos\theta - \tau\sin\theta$$
$$\tau' = x\sin\theta + \tau\cos\theta,$$

where $\tan\theta = iv/c$, plays the part of the phenomenon of light-propagation in Special Relativity.* Although it has been long known that a transformation whereby straight lines become straight lines is affine, very little use seems to have been made of this method of presentation in the " philosophical " expositions of Special Relativity.† The plane of affine geometry has an exceptional line—the line at infinity—and the affine transformations are the collineations which carry the line into itself. The following simple derivation of the Lorentz transformation-formulae, emphasizing the affine character of the transformation, may have some slight advantage over the usual proofs set forth in elementary textbooks.

Reverting to our earlier representation in the *Gazette*, any moving particle (or observer) is specified by means of a space-coordinate x and a time-coordinate t assigned by the observer S and a space-coordinate x' and a time-coordinate t' assigned by the observer S'. Each observer is supposed, as in the classical expositions of Special Relativity, to carry his own appliances for

* Cp. C. W. O'Hara and D. R. Ward, *An Introduction to Projective Geometry* (Oxford, 1937), 12. 6 ; also my approach to Special Relativity in *Mind*, Vol. XLVI, pp. 169–74, pp. 415–16 ; Vol. XLVII, pp. 550–2 ; Vol. LVIII, pp. 218–20.

† See for instance A. P. Ushenko, *The Philosophy of Relativity* (George Allen & Unwin), p. 39 *et seq.*

measuring space and time. Motion with a uniform velocity is described in terms of a linear relation between the space- and time-coordinates. Three assumptions suffice for obtaining the Lorentz transformation-formulae: (1) The velocity of light is uniform and constant for both observers; (2) The relative velocities of S and S' are equal but opposite in sign; (3) Continuity for space-and time-coordinates is preserved in the passage from S to S' (and *vice versa*) by linear transformations which signifies that the transformations are affine and preserve the line at infinity.

From assumption (3) it follows that a linear relation between x and t corresponds to a linear relation between x' and t'. Supposing, for the sake of convenience, that when the observers are together, $t=t'=0$, the affine transformations are

$$x' = \alpha x + \beta t$$
$$t' = \gamma x + \delta t \quad \text{.....(i)}$$

Now by assumption (1), the relation $x=ct$, which signifies the passage of a light-signal emitted when S and S' are together, corresponds to $x'=ct'$. Also $x=-ct$ corresponds to $x'=-ct'$. But, on substitution from (i), $x'=ct'$ becomes

$$(\alpha x + \beta t) = c(\gamma x + \delta t)$$
or, $$(\alpha - c\gamma)x = (c\delta - \beta)t$$

Identifying this and $x=ct$, we get $\alpha - c\gamma = (c\delta - \beta)/c$.

Similarly, on substitution from (i) in $x'=-ct'$ and identifying the resulting relation with $x=-ct$, we get

$$\alpha + c\gamma = (c\delta + \beta)/c.$$

From the two relations

$$\alpha - c\gamma = \delta - (\beta/c) \quad \text{and} \quad \alpha + c\gamma = \delta + (\beta/c)$$

we get $\alpha=\delta$, $c^2\gamma=\beta$. Thus the transformation scheme (i) reduces to

$$x' = \alpha x + c^2\gamma t$$
$$t' = \gamma x + \alpha t \quad \text{.....(ii)}$$

The velocity of the observer S' relative to S is v, so that $dx/dt=v$ when $dx'/dt'=0$.

But
$$\frac{dx'}{dt'} = \frac{\alpha(dx/dt) + c^2\gamma}{\gamma(dx/dt) + \alpha},$$

which gives
$$\alpha v + c^2\gamma = 0.$$

Then (ii) becomes

$$x' = \alpha(x - vt), \quad t' = \alpha(t - vx/c^2). \quad \text{.....(iii)}$$

From this we get

$$\left.\begin{aligned} x' + vt' &= \alpha\{1 - (v^2/c^2)\}x \\ t' + vx'/c^2 &= \alpha\{1 - (v^2/c^2)\}t \end{aligned}\right\} \quad \text{.....(iv)}$$

But on interchanging the roles of (x, t) and (x', t') in (iii) and using assumption (2), we get

$$x = \alpha(x' + vt'), \quad t = \alpha(t' + vx'/c^2). \quad \text{.....(v)}$$

Hence comparing (iv) and (v),

$$\alpha^2(1 - v^2/c^2) = 1.$$

Thus finally we obtain the transformation-formulae in the form

$$x' = \frac{x - vt}{\sqrt{(1 - v^2/c^2)}}, \quad t' = \frac{t - vx/c^2}{\sqrt{(1 - v^2/c^2)}}.$$

C. T. K. CHARI.

2220. *On the Frenet formulae.*

Although the geometrical interpretation of the Frenet formulae is extremely simple, I have not previously seen the diagram described in this note, but I have found it very useful when explaining these equations.

At any point P_0 on a twisted curve Γ where the curvature κ and torsion τ exist, the derivative (with respect to the arc length s) of the unit tangent $\boldsymbol{\lambda}_0$, the unit principal normal $\boldsymbol{\mu}_0$ and the unit binormal $\boldsymbol{\tau}_0$, are given by the Frenet formulae :

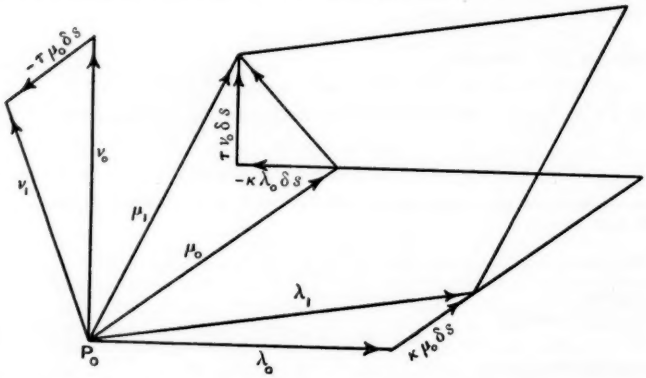
$$\begin{aligned} d\boldsymbol{\lambda}_0/ds &= \kappa\boldsymbol{\mu}_0 \\ d\boldsymbol{\mu}_0/ds &= -\kappa\boldsymbol{\lambda}_0 + \tau\boldsymbol{\nu}_0 \\ d\boldsymbol{\nu}_0/ds &= -\tau\boldsymbol{\mu}_0. \end{aligned}$$

Torsion is taken to be positive for clockwise rotation of the binormal when viewed in the direction of increasing arc length.

At a neighbouring point P_1 of Γ, displaced from P_0 by the infinitesimal arc length δs the unit vectors therefore become

$$\begin{aligned} \boldsymbol{\lambda}_1 &= \boldsymbol{\lambda}_0 + \kappa\boldsymbol{\mu}_0\delta s, \\ \boldsymbol{\mu}_1 &= \boldsymbol{\mu}_0 - \kappa\boldsymbol{\lambda}_0\delta s + \tau\boldsymbol{\nu}_0\delta s, \\ \boldsymbol{\nu}_1 &= \boldsymbol{\nu}_0 - \tau\boldsymbol{\mu}_0\delta s \end{aligned}$$

and by parallel displacement of this triad to P_0 the diagram follows.



The diagram shows why the derivative of the principal normal has two components and gives a picture of how curvature and torsion define the properties of the curve.

T. D. SILLS.

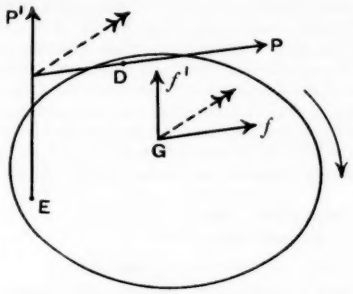
2221. *Resultant of two forces acting simultaneously on a rigid body.*

The fact that the experimental verification of the magnitude and position of the resultant fails in many cases is often overlooked. When the lines of the strings attached at D and E intersect at any given point O outside the body, the line of the resultant can intersect the body only if the ratio P/P' does not lie between two values, for which the resultant is tangential to the body. When P/P' lies between these limits, no point of the body is available at which to attach the third string which would give equilibrium.

The addition of a " light " extension to the body to include the point O, in

order to attach the third string, cannot give a result which is convincing to anyone, for a new circumstance has been introduced.

The assumption of the resultant as an axiom cannot be convincing either; also this is unnecessary.



Consequently, the only proof is a theoretical one. P gives the centre of mass G an acceleration f equal to P/m and parallel to P; P' gives G an acceleration f' equal to P'/m and parallel to P'; hence G has a visible acceleration parallel to the diagonal T of the parallelogram formed by P and P', and equal to T/m. P, P' give the body angular accelerations A, A' about G equal respectively to Pp/mk^2, $P'p'/mk^2$, where p, p' are the perpendiculars from G to the lines of P, P'. Hence the total angular acceleration about G is $(Pp + P'p')/mk^2$. Now by geometry, $Pp + P'p' = Tt$, where t is the length of the perpendicular from G to the diagonal. Hence the angular acceleration is Tt/mk^2, which is the angular acceleration given by T if, and only if, T acts along the diagonal. Hence the body behaves as though it were receiving a force T represented in magnitude and position by the diagonal. H. A. BAXTER.

2222. *On a three-particle problem in elementary dynamics.*

A familiar example on the dynamics of several particles, to be found in a number of textbooks, is as follows. "Particles A, B and C, of equal mass, rest at the corners of an equilateral triangle on a smooth horizontal table. AB and BC are light, inextensible strings. If A is given a velocity v in the direction CB, show that when the string AB tightens again, C starts to move with velocity $v/15$." This is intended as an example on angular momentum and one is evidently required to assume that the strings remain taut after the tightening of AB, so that the velocities are as shown in Fig. 1.

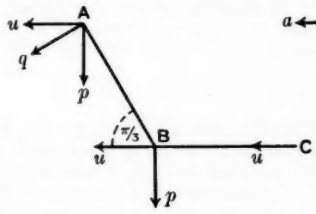


FIG. 1

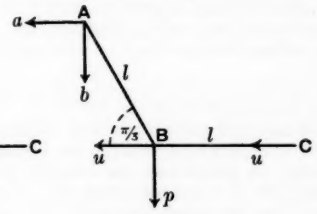


FIG. 2

By writing down the three equations provided by the principles of con-

servation of momentum and angular momentum, it is easily verified that the assumption leads to the result stated.

It may be shown, however, that the assumption is inconsistent with the dynamics of the system, for a further condition is provided by the conservation of energy (in this case wholly kinetic). Using this instead of the angular momentum equation, it is found that the velocity of C is given by a quadratic equation which has no real roots. It is thus clear that one or both strings must slacken, and therefore that one must insert arbitrary values for the velocity components of the particles. There are then five unknowns, and since the conservation laws provide four equations, the problem is indeterminate.

One can get a determinate problem by replacing the string BC by a light rod. The scheme of velocities is then given by Fig. 2, in which there are only four unknowns. Taking angular momenta about A, the conservation laws give the equations

$$mv = 2mu + ma,$$
$$0 = mb + mp,$$
$$0 = -2mul \sin \tfrac{1}{3}\pi - mpl \cos \tfrac{1}{3}\pi,$$
$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m(a^2+b^2) + \tfrac{1}{2}m(u^2+p^2) + \tfrac{1}{2}mu^2,$$

from which it is found that $u=0$ or $u=2v/15$. The fact that the string is inextensible requires that the solution $u=0$ be rejected, and the string does, in fact, slacken.

The assumption in the original problem could be justified by adding the condition that the strings "freeze" at the instant AB tightens. (This artifice is used in the elementary theory of the catenary.) The solution given is then correct and one is faced with a more difficult task in compromising the energy and angular momentum equations. The answer is evidently that work is done by the "freezing". The solution of the problem and the energy equation may then be actually used to calculate this work, and it is found to be $7mv^2/20$. It is not obvious, however, what the forces can be that do this work, since they would seem to interfere with the general rules regarding light strings and rods and the fact that the internal forces in them do no work. It becomes clear if one supposes a slackening string to stiffen in a small but finite time δt. The stiffening causes an impulsive force and since the string buckles slightly, work is done. If δt is reduced, the buckling is less but the average force is greater, and it is reasonable to assume that as $\delta t \to 0$, the work tends to a non-vanishing limit.

The instantaneous stiffening of the string causes an infinite impulse, and the need for the above explanation is an example of the difficulty one can get into through treating "infinity" as if it were a number. One can not assume that a set of rules holding for certain entities still hold when one introduces an infinity, for the infinite process only has meaning, and therefore can only be properly inderstood, as the limiting case of a finite process, and in the finite process certain entities may be changed in such a way that the rules do not hold.

R. CADE.

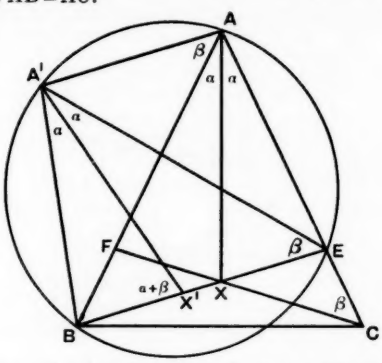
2223. *Variations on an old theme.*

An old and well-known problem in pure geometry is to prove that if two of the angle bisectors of a triangle are equal then the triangle is isosceles.

This may be generalised as follows. Suppose that E, F are points on the sides AC, AB of a triangle, that $BE=CF$ and that X is their point of intersection. Then we may prove that $AB=AC$ provided that the position of X is appropriately restricted. The proof is simple if X is, for example, the orthocentre, the circumcentre or the centroid, and rather less simple if X is the incentre as in the famous old problem already mentioned. These restrictions

are, however, unnecessarily narrow, as X may be given more freedom ; it may, for example, be any point on a median or any point on an angle bisector. A proof is given for the latter.

In the triangle ABC, if X is any point on the bisector of the angle A, and $BXE = CXF$, then $AB = AC$.



The point A' is on the circle ABE, such that $\angle BEA' = \angle FCA = \beta$. The triangles AFC, $A'BE$ are then congruent and their angle bisectors AX, $A'X'$ are equal.

$$\angle A'AB = \angle A'EB = \angle ACF = \beta,$$
$$\angle A'X'B = \angle X'A'E + \angle X'EA' = \alpha + \beta = \angle A'AX.$$

Hence A', A, X, X' are concyclic and as $AX = A'X'$ it is easily seen that AA' is parallel to XX' and therefore that the triangles $A'BE$, AEB are congruent. Thus the triangles AFC, AEB are congruent, and thence $AB = AC$.

A. C. COSSINS.

2224. *Proof of a formula in dynamics.*

For a rigid body moving in two dimensions, the moment of force about the instantaneous centre C is L_C, where

$$L_C = \frac{1}{2\omega}\frac{d}{dt}(I_C\omega^2).$$

Let P be any particle of the body, G the centre of mass ; then the system of forces acting on the body can be replaced by a resultant $\Sigma\mathbf{F}$ at G together with a couple $\Sigma\mathbf{r}\wedge\mathbf{F}$. Further, if M is the total mass and f the acceleration of G, $\Sigma\mathbf{F} = M\mathbf{f}$.

The sum of the moments about C is

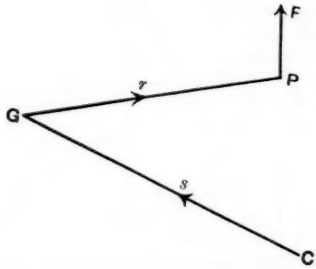
$$\begin{aligned}\mathbf{L}_C &= \Sigma\mathbf{r}\wedge\mathbf{F} + \mathbf{s}\wedge(\Sigma\mathbf{F})\\ &= \Sigma\mathbf{r}\wedge\mathbf{F} + \mathbf{s}\wedge M\mathbf{f}.\end{aligned}$$

But $\Sigma\mathbf{r}\wedge\mathbf{F}$ is the sum of the moments about G and is known to be $I_G\dot{\omega}$, where I_G is the moment of inertia about G and ω is the angular velocity of the body (more precisely, $\omega\mathbf{k}$, where $\mathbf{k}$ is a constant unit vector, normal to the plane).

At the instant G is describing a circle centre C with angular velocity ω and its speed is $s\omega$, where s is the distance CG. Also at the instant CG is the normal to the path of G and the acceleration along the tangent to its path is $d(s\omega)/dt$. Hence the moment of $M\mathbf{f}$ about C is given by

$$Msd(s\omega)/dt = Ms^2\dot{\omega} + Mss\dot{}\,\omega.$$

Hence
$$L_C = I_G\dot{\omega} + Ms^2\dot{\omega} + Ms\dot{s}\omega$$
$$= \frac{1}{2\omega}\frac{d}{dt}\{(I_G + Ms^2)\omega^2\}$$
$$= \frac{1}{2\omega}\frac{d}{dt}(I_C\omega^2).$$



It should be noticed that the acceleration of G along the tangent to its path is not $s\dot{\omega}$, because the point which is the instantaneous centre is changing from instant to instant and s is changing with the time. The expression $s\omega$, however, represents the speed of G at any instant whatsoever provided we take s as the distance from G to the point which is the instantaneous centre at that instant.

When $\dot{s} = 0$, the formula reduces to $L_C = I_C\dot{\omega}$. In this case s is constant, and the locus of C in the body is a circle with centre G. Such a case occurs when for example a circle or circular cylinder or sphere rolls along a straight line, or when one circle rolls inside or outside another circle. H. K. POWELL.

2225. *Geometrical aspects of a linear combination of equations.*

1.1. Consider the equations of n curves, written for brevity as follows:
$$F_1 = 0, \quad F_2 = 0, \ \dots, \quad F_\nu = 0, \ \dots, \quad F_n = 0.$$
Let
$$\Sigma_n = \mu_1F_1 + \mu_2F_2 + \dots + \mu_\nu F_\nu + \dots + \mu_nF_n,$$
where $\mu_1, \mu_2, \dots, \mu_n$ are constants, and let Σ_p be the sum of p such terms as $\mu_\nu F_\nu$ arbitrarily chosen from the above n terms of this type. Let Σ_q be the sum of the remaining terms, so that
$$p + q = n, \quad \Sigma_p + \Sigma_q = \Sigma_n.$$

1.2. The curve $\Sigma_n = 0$ passes through a " group of points " consisting of all the intersections (real and imaginary) of the curves $\Sigma_p = 0$ and $\Sigma_q = 0$. For if the coordinates of a point are such that $\Sigma_p = 0$, $\Sigma_q = 0$, then $\Sigma_p + \Sigma_q = 0$.

1.3. The number of groups, for a given p and q, is
$${}^nC_p = {}^nC_q = \tfrac{1}{2}({}^nC_p + {}^nC_q);$$
hence for all possible curves $\Sigma_p = 0$ and $\Sigma_q = 0$ we have a total
$$\tfrac{1}{2}({}^nC_1 + {}^nC_2 + \dots + {}^nC_{n-1}) = \tfrac{1}{2}\{2^n - ({}^nC_0 + {}^nC_n)\}$$
that is,
$$2^{n-1} - 1 \text{ such groups.}$$

The application of this simple observation, even in the most elementary

cases, enables us to elucidate many interesting geometrical properties, as the following examples show.

1.4. Let us find the number N_n of curves Σ_n formed when each μ_ν can have the value $+1$ or -1. If we form all the curves for $n=\nu$ then we can get the curves for $n=\nu+1$ by adding $+F_{\nu+1}$ or $-F_{\nu+1}$ to each one of the first set, that is, we double the number of curves. Hence

$$N_n = 2N_{n-1} = 2^2 N_{n-2} = \ldots = 2^{n-1},$$

since $N_1 = 1$.

2.1. Suppose that

$$F_\nu = E_\nu = x\cos\theta_\nu + y\sin\theta_\nu - p_\nu.$$

Every curve Σ_n will be a straight line. Two straight lines intersect in one point only. We have, therefore, $2^{n-1}-1$ " points " rather than " groups of points ".

2.2. We know that the distance of a point (x, y) from the line $E_\nu = 0$ is simply E_ν. Hence, for any point situated on either bisector of the angles formed by two lines $E_1 = 0$, $E_2 = 0$ we have

$$|E_1| = |E_2|, \text{ that is, } E_1 \pm E_2 = 0.$$

The bisector $E_1 + E_2 = 0$ is situated in that part of the plane for which E_1 and E_2 have opposite signs, since for a point on $E_1 + E_2 = 0$ for which E_1 is positive, E_2 must be negative. The other bisector of the two lines is given by the conjugate equation $E_1 - E_2 = 0$.

We can suppose that $E_1 = 0$ and $E_2 = 0$ are the equations of two sides of a polygon. We can suppose, further, that the portion of the plane enclosed by the polygon is negative with respect to all the sides, that is, that the origin is situated inside the polygon. Then we can term the bisector $E_1 + E_2 = 0$ " external " and the bisector $E_1 - E_2 = 0$ " internal ".

2.3. To start with a very simple application, let $E_1 = 0$, $E_2 = 0$, $E_3 = 0$ be the equations of the sides of a given triangle. Consider, for example, the equation $E_1 + E_2 + E_3 = 0$. On this line lie the intersections of $E_1 = 0$ with $E_2 + E_3 = 0$, of $E_2 = 0$ with $E_3 + E_1 = 0$ and of $E_3 = 0$ with $E_1 + E_2 = 0$. This is merely a statement of the fact that the external bisectors of the angles of a triangle cut the opposite sides respectively in three points lying on the straight line $E_1 + E_2 + E_3 = 0$.

2.4. Let us now deal with a quadrilateral. Since $2^{4-1} = 8$, there are eight lines $\Sigma_n = 0$ which can be considered, namely,

$$\pm E_1 \pm E_2 \pm E_3 \pm E_4 = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Take the line $E_1 + E_2 + E_3 + E_4 = 0$, say. On it lie the intersections of the following lines :

$$\left.\begin{aligned} &E_1 = 0,\ E_2 + E_3 + E_4 = 0\ ; \quad E_2 = 0,\ E_3 + E_4 + E_1 = 0 \\ &E_3 = 0,\ E_4 + E_1 + E_2 = 0\ ; \quad E_4 = 0,\ E_1 + E_2 + E_3 = 0 \end{aligned}\right\}, \quad \ldots\ldots\ldots\text{(ii)}$$

and

$$\left.\begin{aligned} &E_1 + E_2 = 0,\ E_3 + E_4 = 0 \\ &E_2 + E_3 = 0,\ E_4 + E_1 = 0 \\ &E_1 + E_3 = 0,\ E_2 + E_4 = 0 \end{aligned}\right\}. \quad \ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

The equations (ii) state that the lines determined by the feet of the external bisectors of a triangle formed by any three sides cut the remaining sides respectively in three collinear points.

The pairs of equations (iii) represent a new property : in any quadrilateral the points of intersection of the external bisectors of the opposite angles are collinear.

Similar properties are obtainable from the other equations (i). Bearing in mind the convention adopted in 2.2, we can select at once all the combinations of bisectors which exhibit similar properties in a given quadrilateral. A proof based on traditional geometrical arguments can be established in each case, but such methods would hardly give a general criterion for finding, *a priori*, what combinations of bisectors give collinear points.

2.5. The number of lines and of points on each line increases rapidly with the number of sides. For example, for an icosagon we find 524,288 lines and on each of them 524,287 points, that is, we find in this case 274,877,382,656 points disposed in groups on straight lines.

2.6. We now have enough information to enable us to examine geometrically a problem of the following type : given three lines $E_1=0$, $E_2=0$, $E_3=0$, let α, β, γ be the distances of any point in the plane from each line respectively ; to find the locus of points which satisfy the relation $\alpha=\beta+\gamma$.

The locus obviously consists of the segments of the lines

$$+E_1+E_2+E_3=0,$$
$$-E_1+E_2+E_3=0,$$
$$+E_1-E_2+E_3=0,$$
$$+E_1+E_2-E_3=0,$$

where $|E_1|=|E_2|+|E_3|$.

3.1. Suppose that

$$F_\nu=K_\nu=x^2+y^2-2xx_\nu-2yy_\nu-c_\nu.$$

Instead of "distances" we can speak here about "powers" and easily obtain results similar to the previous case. But let us examine more closely a particular case which clarifies the application of our principles and also gives us a new circle of the triangle.

First let us fix the position of the circle $\mu_1K_1+\mu_2K_2=0$. We know two points on this circle. Also it is evident that the centre of this circle is the point

$$\left(\frac{\mu_1x_1+\mu_2x_2}{\mu_1+\mu_2},\ \frac{\mu_1y_1+\mu_2y_2}{\mu_1+\mu_2}\right)$$

which is situated on the line of centres of $K_1=0$ and $K_2=0$ and divides this line in the ratio $\mu_2:\mu_1$. If $\mu_1+\mu_2=0$ we have the radical axis of $K_1=0$ and $K_2=0$.

In the same way the centre of $mK_1+nK_2+rK_3=0$ is the point

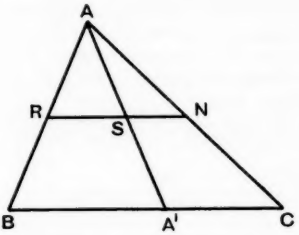
$$K=\left(\frac{mx_1+nx_2+rx_3}{m+n+r},\ \frac{my_1+ny_2+ry_3}{m+n+r}\right)$$

that is, the point K has m, n, r as coordinates with reference to the triangle of centres of the given circles.

3.2. Let N and R be the centres of the circles $K_B=0$, $K_C=0$ drawn respectively on the sides AC and AB of a triangle as diameters. The centre of $nK_B+rK_C=0$ is at the point S on NR such that $RS/SN=n/r$. Let A' be the point of intersection of AS and BC. Then $RS/SN=n/r=BA'/A'C$. But $AS=A'S$; therefore AA' is a diameter of the circle $nK_B+rK_C=0$. Similarly we can show that the circles $rK_C+mK_A=0$ and $mK_A+nK_B=0$ have as diameters the lines BB' and CC' such that $CB'/B'A=r/m$, $AC'/C'B=m/n$.

Hence $$BA' \cdot CB' \cdot AC' = A'C \cdot B'A \cdot C'B;$$

that is, the cevians AA', BB', CC' cut each other in a single point whose coordinates are $(1/m, 1/n, 1/r)$ for the triangle of reference ABC.



We can now state the following interesting property : in a triangle the six points of intersection of the circles drawn on its sides as diameters with those which are drawn respectively on three concurrent cevians lie on a circle. Further, the centre of this circle possesses the same position in the anti-complementary triangle as the inverse of the intersection of the cevians in the original triangle.

4. These considerations can also be applied to space. If the word " surface " is substituted for the word " curve " and the word " curve " for the word " point ", a great part of the above text can be repeated without further proof. Thus, the planes which bisect externally the angles of a trihedral angle cut the opposite sides in three coplanar lines. The spheres which have the sides of a triangle as diameters cut those of the medians respectively in three circles which lie on a new sphere with its centre at the centroid of the triangle.

CDR. G. N. VLAHAVAS, R.H.N.

2226. *The luck of the toss in squash rackets.*

In the game of squash a player wins a point if he serves and wins a rally, but not if he wins a rally in which he did not serve ; the winner of a rally serves for the next. A game is won by the first player to obtain nine points (ignoring the slight change which may be made at 8-all). The question arises of the advantage given to the player who takes the first service, supposing the players to be of equal ability.

We take the phrase " of equal ability " to mean that in any rally A's chances of winning are $\frac{1}{2}$ and B's are $\frac{1}{2}$. When A needs m points and B needs n points to win the game, call A's chances of winning the game $a_{m,n}$ when A is serving and $b_{m,n}$ when B is serving. We have immediately

$$\left.\begin{aligned} a_{m,n} &= \tfrac{1}{2}(a_{m-1,n} + b_{m,n}) \\ b_{m,n} &= \tfrac{1}{2}(a_{m,n} + b_{m,n-1}) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

with the initial conditions $a_{0,n} = 1$, $b_{m,0} = 0$ $(m, n \geqslant 1)$. The symbols $a_{m,0}$, $b_{0,n}$ are undefined.

Using the generating functions

$$A \equiv A(u, v) \equiv \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} u^m v^n,$$

$$B \equiv B(u, v) \equiv \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{m,n} u^m v^n,$$

and multiplying each of the equations (1) by $u^m v^n$ and summing, we have by virtue of the initial conditions

$$(1-\tfrac{1}{2}u)A=\tfrac{1}{2}uv/(1-v)+\tfrac{1}{2}B,$$
$$(1-\tfrac{1}{2}v)B=\tfrac{1}{2}A,$$

so that on reduction

$$A=\frac{\tfrac{1}{2}uv(1-\tfrac{1}{2}v)}{(1-v)\{(1-\tfrac{1}{2}u)(1-\tfrac{1}{2}v)-\tfrac{1}{4}\}},$$
$$B=\frac{\tfrac{1}{4}uv}{(1-v)\{(1-\tfrac{1}{2}u)(1-\tfrac{1}{2}v)-\tfrac{1}{4}\}}.$$

Hence

$$A-B=\frac{\tfrac{1}{4}uv}{(1-\tfrac{1}{2}u)(1-\tfrac{1}{2}v)-\tfrac{1}{4}}.$$

Thus

$$A-B=\tfrac{1}{4}uv\sum_{r=0}^{\infty}(1-\tfrac{1}{2}u)^{-(r+1)}(1-\tfrac{1}{2}v)^{-(r+1)}(\tfrac{1}{4})^r,$$

in which the coefficient of $u^m v^n (m, n\neq 0)$ is

$$a_{m,n}-b_{m,n}=(\tfrac{1}{2})^{m+n}\sum_{r=0}^{\infty}\left\{\frac{(m)_r(n)_r}{r!\,r!}(\tfrac{1}{4})^r\right\},$$

so that

$$a_{m,n}-b_{m,n}=(\tfrac{1}{2})^{m+n}F(m,n;\ 1;\ \tfrac{1}{4}). \qquad \text{.....(2)}$$

Since the players are of equal ability, we also have the relation

$$a_{m,n}+b_{n,m}=1 \qquad \text{.....(3)}$$

and it follows from (2) and (3) that

$$a_{m,m}=\tfrac{1}{2}+(\tfrac{1}{2})^{2m+1}F(m,m;\ 1;\ \tfrac{1}{4}).$$

Replacing * $F(m,m;\ 1;\ \tfrac{1}{4})$ by $(\tfrac{4}{3})^{2m-1}F(1-m,1-m;\ 1;\ \tfrac{1}{4})$, we have in the case $m=9$

$$a_{9,9}=\tfrac{1}{2}+\tfrac{1}{4}(\tfrac{8}{3})^{17}F(-8,\ -8;\ 1;\ \tfrac{1}{4})$$

which may be calculated, since the series terminates. We obtain

$$a_{9,9}=\frac{7{,}674{,}706}{14{,}348{,}907}.$$

H. Ap Simon.

2227. *Right-angled triangles.*

Arising from Mr. B. D. Price's article in the *Gazette* for May, 1949, can any member assist in finding integral right-angled triangles, each of whose sides is of the form s^2-1. For example, in the case $x=7$, $y=4$ the triangle sides are 64, 56, 33. If each of these sides is multiplied by the common factor 13, the derived triangle has sides 195, 168, 99, that is, 14^2-1, 13^2-1, 10^2-1.

The question can be put generally: find integral right-angled triangles which satisfy the following conditions:

$$A^2-1=p(x^2+y^2),$$
$$B^2-1=p(x^2-y^2),$$
$$C^2-1=p(2xy).$$

* W. N. Bailey, *Generalized Hypergeometric Series*.

The problem was originally propounded by Mr. J. Travers. A second triangle which fulfils the conditions is known. It is now desired to know whether there are any others, or alternatively to learn of a method of calculation to determine all possible solutions. J. K. BAILEY.

2228. *Constructions for the roots of a quadratic equation.*

It is unlikely that either of the following constructions is original, but for those readers who do not know them they may be of interest.

1. The first is merely an adaptation of the method given on p. 283 of Professor H. G. Forder's *School Geometry*, making it possible to find complex roots by the same construction. For real roots, the circle on the segment (0, 1), (a, b) as diameter meets the axis of x in points whose abscissæ are the roots of the equation $x^2 - ax + b = 0$.

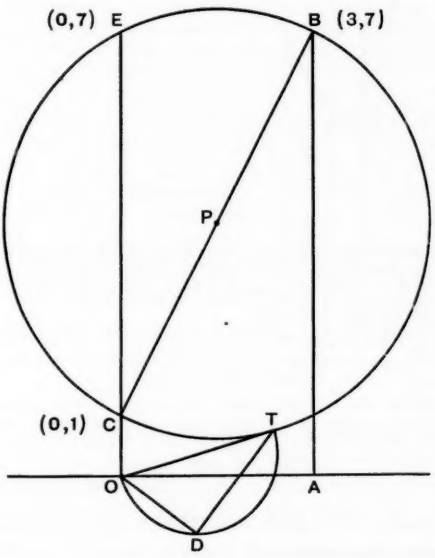


FIG. 1

The figure illustrates the graphical solution of $x^2 - 3x + 7 = 0$, whose roots are $\frac{1}{2}(3 \pm i\sqrt{19})$. The circle does not meet the axis of x. In this case, we draw the circle whose diameter is the tangent OT, and then the chord OD of this circle, of length $\frac{1}{2}a = \frac{3}{2}$. The join DT gives the magnitude of the imaginary part of the roots; the real part is of course OD.

For: $OT^2 = OC \,.\, OE = 7,$

$TD^2 = OT^2 - OD^2 = 19/4.$

Thus $x = OD \pm TD \,.\, i.$

2. The second method is quite independent of this, but gives the complex roots as vectors in the Argand diagram.

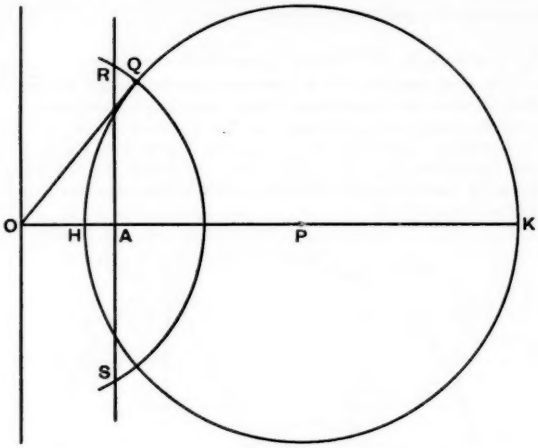


FIG 2

For the same equation, construct the circle on the segment HK (1, 0), (7, 0) as diameter, draw the tangent OQ from the origin, and erect the perpendicular AR at A where $OA = \frac{3}{2}$. Then describe the arc of radius OQ to cut AR at R, S. The roots are the vectors OR, OS.

For: if the roots are z_1, z_2, then $z_1 z_2 = 7$. But the roots are conjugate, $z_2 = \bar{z}_1$, and hence

$$z_1 z_2 = z_1 z_1 = |z_1|^2 = 7.$$

Thus $OQ = |z_1| = |z_2|$.

Also $z_1 + z_2 = 3$, but since $z_2 = \bar{z}_1$, $OA = \mathbf{R}(z_1) = \mathbf{R}(z_2) = 32$.

Thus $OR = z_1$, $OS = z_2$. J. H. CLARKE.

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of those sending the questions will not be published.

Applicants are requested to return all solutions to the Secretary of the Bureau.

REVIEWS.

1. **Die Entwicklungs-Geschichte der Leibnizschen Mathematik.** By J. E. HOFMANN. Pp. 252. 1949. Leibniz Verlag Munchen. (Bischer R. Oldenbourg Verlag)

2. **Leibniz zu seinem 300. Geburtstag (1646-1946)** Herausgegeben von E. HOCHSTETTER. Lieferung 4, Leibniz' Mathematische Studien in Paris. By J. E. HOFMANN. Pp. 70. 1948. (Walter de Gruyter, Berlin)

Of these two works, which are intimately connected, the former is a sustained account of the mathematical development and discoveries of Leibniz, with particular reference to what he accomplished during his four years in Paris, 1672–1676. The central theme is the discovery of the calculus, the whole situation being very carefully studied. This includes a full and historical account of what led up to it, covering the middle seventeenth century from the time of Fermat and Descartes, and an evaluation of what others achieved among both the forerunners and contemporaries of Leibniz. The larger work contains the relevant mathematical results, and in fact the whole account and argument, in considerable detail: the smaller is essentially an abridgement of the larger, obtained by omitting most of the technical details. It is a work appropriate to the occasion, that of a 300th anniversary in honour of Leibniz' birthday in 1646, and it is meant for a wider circle than that of mathematical readers. It is the fourth of a set of special contributions, being preceded by those which are concerned with Leibniz as metaphysician (Nicolai Hartmann), Leibniz and Peter the Great (Ernst Benz), and a Leibniz Memorial (Erich Hochsetter), and it is followed by a fifth contribution on Leibniz as logician (Karl Dürr). This shorter work retains much of the biographical, psychological and philosophical treatment that is to be found in the larger work. Both are very readable and illuminating, for the subject, that of the discovery of the calculus, is still something of a mystery. Among those who took part in the drama of this discovery Barrow, Gregory, Newton and Leibniz stand out pre-eminently: and the problem turns on what it was that each of these men of genius discovered, and how far each was dependent on another. The mystery is heightened by the rapidity with which two of these, Newton and Leibniz, reached to the heart of the matter: Newton in his early postgraduate years at Cambridge and Leibniz ten years later at Paris; Newton starting from the tutelage of Barrow, and Leibniz from that of Huygens.

Professor Hofmann has told the story of this astonishing development of Leibniz' mathematical powers with care and insight. He makes his hero live, and he draws largely and systematically for his material from contemporary documents and letters. His conclusion is as follows: that Leibniz had a better and sharper understanding of the fundamentals than had any of his contemporaries and that he is to be held as the exclusive inventor of the calculus: and that in this creation he was influenced by no one else except in the most trifling details. It seems to the present reviewer that this extreme view hardly follows from the facts that the author has so carefully stated, and that the effect of the work would have been heightened had the author come to a more cautious conclusion.

Both works contain an excellent list of letters arranged chronologically, together with the sources where they may be found. These letters range over the names of Descartes, Fermat, Wallis, Brouncker, Collins, Gregory, Barrow, Oldenburg, Leibniz, Huygens, Sluse, Newton, Tschirnhaus, Bernoulli and others. As the narrative proceeds, each statement is carefully referred to one or other letter: and in many cases the substance of a whole

letter is paraphrased. To judge from the cases where the original letter can be consulted this paraphrasing has been very well done. Thirty and more years ago Professor Gibson and Sir Edmund Whittaker drew attention to the important collection by Rigaud of *Correspondence of Scientific Men in the Seventeenth Century*, based upon letters preserved at Shirburn Castle in the possession of the Earl of Macclesfield. This has led to a closer study of the correspondence that passed between Collins and Newton and Gregory, and this has helped to clear up several obscurities that had a wide currency as well as to establish important facts that had been forgotten or ignored. This in its turn led to the discovery of letters and manuscripts of James Gregory, principally in the University Library at St. Andrews, which proved him to be in the front rank among the immediate predecessors of Liebniz as a mathematical discoverer. What distinguishes the work of Hofmann from the many notable contributions in the past, by Cantor and others, is the acceptance of the evidence of this new material and of the importance of Gregory : for example, that during the years 1668–71 Gregory had found " a universal method ", as he called it, which included the general binomial expansion and also complicated instances of successively derived functions leading to Taylor expansions of a function in the form of a power series, and also the fundamental expansion theorem of interpolation. Hofmann judges that in this work Gregory was essentially independent of Newton. This is to go further than Gibson probably would have gone, and it is a very interesting feature. One may recall that when the *Commercium Epistolicum* (1712) was published it gave an official verdict by the Royal Society that not Leibniz but Newton was the first inventor of the method : " that the differential method is one and the same with the method of fluxions excepting the name and mode of notation." The notes that accompany the documents and letters in this *Commercium* do no small injustice to Leibniz, while certain imperfect quotations do the same for Gregory in a lesser degree. The examination of Leibniz' own manuscripts and at a much more recent date those of Gregory has now made it possible to paint a truer picture of what actually took place. Hofmann draws attention to the very early date of Gregory's invention of the binomial theorem (1668), but does not agree with the long held, but recently questioned, view that Newton had invented the same series three years earlier still. The doubt was occasioned by a misinterpretation of a passage in Newton's letter to Leibniz (24th October, 1676) which actually describes the invention as one of his earliest (1665). This point is only important in suggesting that to arrive at the facts an equal care must be taken in the study and evaluation of the work of each actor in this great story.

Professor Hofmann goes into the question whether Leibniz derived any help from Newton and Gregory, and places before the reader the material which might supply this help. He concludes that any such help is slight ; and he bases an interesting and persuasive argument on the actual notes which Leibniz made on receiving the material. Thus the existence of a note upon an item of Gregory in the theory of numbers, while another item, on analysis, is passed over, suggests that Leibniz was alert to what was novel and indifferent to what was already familiar to him. The possibility that Leibniz drew some inspiration from Barrow's *Lectiones Geometricae*, a book which he bought in London (1673) and took to Paris, is mentioned but dismissed, perhaps too lightly in view of the serious argument of J. M. Child to the contrary. In the *Lectiones* which were published in 1670 Barrow gave a systematic set of geometrical theorems which when put into algebraic notation reveal the now ordinary laws of differentiating sums, differences, products, quotients and more. A few contemporaries, among whom were Gregory and Sluse, read the book with discernment : admittedly it was a hard book for

most. The argument turns on whether men found it hard because it was written in a geometrical idiom or because it involved strangely new ideas. J. M. Child adduces arguments that make it difficult to believe that Leibniz was uninfluenced by this book.

In both these studies by Hofmann it is specially interesting to read the early part of the story of Leibniz' mathematical development, where the leaps from particularities to general mathematical principles are exhibited. The beautiful passage from Pascal's arithmetical triangle to the harmonical triangle (where the entries are reciprocals of binomial coefficients) is highly original and gives convincing insight into Leibniz' analytical power. Much is also told about the theory of equations and Leibniz' considerable work in algebra. This broad survey, including many details, brings into accessible form much that can otherwise only be traced with difficulty. The enigmatical Pell appears upon the scene, and the bold attempt of Gregory to establish a resolvent for a quintic equation is included in the story.

Each of these books is a work of long and devoted scholarship, and is a mine of important information upon the history of a peculiarly significant generation when mathematics advanced by its greatest leap since the time of the Ancient Greeks. Behind this study is a detailed and by no means unsympathetic examination of the work of mathematicians in England and Scotland, and particularly of Newton and Gregory. It would seem that in his conclusion, already mentioned, that Leibniz should be considered the sole inventor of the calculus, the author is expressing and accepting the verdict of Leibniz himself, at any rate as matters appeared to him in the early days before the controversy on priority arose. In arriving at this verdict the author has insisted on the particular examples that Newton and Gregory vouchsafed, as interesting and skilful, but not such as can be compared with the general fundamentals as given and held by Leibniz. At the same time the author welcomes the proposal of the Royal Society to publish letters and manuscripts that have hitherto lain hid: a step that will certainly help to clarify the issue. As an example of this there is now documentary evidence to show that when Newton said he had discussed the curvature of a plane curve at a very early date he meant what he said. In May 1665 he expressed the coordinates of the centre and the radius of curvature explicitly in terms of the first and second partial derivatives with regard to x and y of the algebraic expression which when equated to zero gave the nature of the curve.

Professor Hofmann has done a great service to the mathematical world by his vivid portrayal of Leibniz in what is after all only one aspect of this versatile man of genius, and these books will be gratefully read by all who value the history of their science.

H. W. TURNBULL.

Mathematics and the Imagination. By EDWARD KASNER and JAMES NEWMAN. With drawings and diagrams by Rufus Isaacs. Pp. xiv, 380. 15s. 1949. (Bell)

The final sentence sums up very well the scope of this interesting book: "For in their prosaic plodding both logic and mathematics often outstrip their advance guard and show that the world of pure reason is stranger than the world of pure fancy." In some ways the book is very like Rouse Ball's *Mathematical Recreations and Essays*, but it is obviously more ambitious. It is packed with information and deals with all the elementary ingredients of mathematics, number, shape, line, series, pattern, infinity, paradox, chance, change, with copious and entertaining visual aids: and the treatment is refreshing. The book is at once a collection of mathematical puzzles and surprises, a pleasantly written history of mathematical thought, and a real attempt to bring before the interested but unprofessional reader the charac-

teristic findings of mathematicians. The book is enlivened by an excellent introduction, promising good things to come, and by a pertinent quotation at the head of each chapter taken from Chaucer, Heraclitus, Descartes, Pascal, Mark Twain, Gilbert (not the discoverer of the magnet but the playwright), Eddington, popular advertisement, Spenser, Bertrand Russell, William James and Lewis Carroll. These chapters are not numbered: they just occur. This gives a hint of improvisation, which in a book dealing with exact reasoning is part of its charm. Their numbers are stated in the table of contents, and they are followed by an excellent index and bibliography. The titles of the chapters are interesting: "New Names for Old" is an essay on nomenclature where the innocence of such homely words as group, ring, simple curve, limit is contrasted on the one hand with the methylpropenylenedihydroxycinnamenylacrylic acid of the chemist and on the other hand with the disconcerting depth of exact meaning which mathematicians extract from these words.

There is a very racy account of large numbers and of infinity: 10^{100} is called a googol, a name invented in a kindergarten school, but the number 10^{11} is not called a billion. The estimate that the number of words spoken, "including all baby talk, love songs, and Congressional debates", totals about ten million billion has to be taken with reserve until one recalls that in America a billion is but a thousandth part of what it is in Britain! Figure 2, that displays a Jordan curve as a panel of futuristic art in contrast to the simplicity of Figure 1, a lemniscate, is a thing of delight. "Beyond the Googol"—the next chapter—tells of Cantor and of Zeno. Next, the chapter "Transcendental and Imaginary", starts with the hybrid pun "pie (π, i, e)" and wanders refreshingly through the fields of number. Analytical is related to Greek geometry as programme music to absolute music. "Assorted Geometries—Plane and Fancy" follow in the next chapter, which involves dimensions, mappings, and non-Euclidean geometries. Then "Pastimes of Past and Present Times", a chapter which includes Chinese rings, scales of notation, the fifteen puzzle and other well-known friends. "Paradox Lost and Paradox Regained" deals with arithmetical and geometrical fallacies, optical illusions, and logical paradoxes. "Chance and Chanceability" is a timely and provocative chapter. "Rubber-Sheet Geometry" follows with Euler's topological excursions over the bridges at Königsberg and further examples of Jordan curves (or closed circuits), which return to the pleasantries of the opening chapter. Connectivity and the four-colour problem find their place here, as also a three-colour problem that has been proposed and solved by L. J. Brouwer to this effect: that a rectangular map can be imagined which consists of three countries, each bounded by a continuous frontier, where each country is simply connected, and the map is entirely filled by these three countries, and yet where every meeting point of two countries is a meeting-place of all three countries!

"Change and Changeability—The Calculus", is an adequate but more usual type of chapter, until its Appendix is reached, which introduces "Pathological Curves". The Snowflake Curve is the limit attained by repetition of a process starting with an equilateral triangle, a closed figure of three sides. Each side of the figure is to be trisected, and on each middle segment as base an equilateral triangle is then constructed outwards; whereupon the base is removed. This gives a 12-sided star-shaped figure: and the process is repeated on each side, giving a 48-sided figure, and so on. The perimeter of the limiting form of this polygon lies within a finite area and its area is finite; yet the perimeter is infinite, and the polygon is a Jordan curve. Other curiosities are the in-and-out curve and the space-filling curve. The criss-cross curve is one which crosses itself at every point: unfortunately the

explanation of this curve is tantalisingly imperfect (in an otherwise beautifully lucid book). Probably a sentence has got left out in error. The directions run : (1) inscribe a triangle within a triangle ABC and shade the interior triangle (giving one shaded and three unshaded triangles) : (2) continue the process for each unshaded triangle (giving in all nine unshaded and four shaded triangles) : (3) to (n), repeat the process indefinitely. " Then join the points of the original triangle remaining unshaded and distort the original triangle so that the three points A, B and C are brought together. There you have the Crisscross Curve." The word " join " is ambiguous : presumably one joins a point of an unshaded small triangle (which has shrunk indefinitely) not to *any* other such point elsewhere but only to such a point of a contiguous unshaded triangle : then the totality of these joins will give the required curve.

The final chapter, on Mathematics and the Imagination, deals with the question, what is mathematics? The chapter is short, philosophical and interesting. Occasionally statements of personal views go too far, as, for instance, in assuming that all present-day mathematicians have given up the notion that mathematical truths have an existence independent of and apart from our own minds. Some shrewd words are said for and against logical positivism : " though we have scourged ourselves pitilessly and driven out the confusing spirit of metaphysics may we also have drained the vitality of mathematics? May we not well have lost ' the spirit of the word '? " This is well said.

To a mathematician the book is most enjoyable : but I wanted to know how it struck a friend, and this is what he, an actuary by profession, said : " I started to read the book in the hope of learning a little of what advanced mathematics is really about. A lot of people like engineers and actuaries who learn a certain amount of utilitarian mathematics must wonder, as I do, what people like yourself spend your time working on. This book does seem to give a few glimpses, but I was left with an unsatisfied feeling. . . . The key sentence of the book seems to be that ' the world of pure reason is stranger than the world of pure fancy '. The authors do seem to put up a strong case in support of that thesis : and they do bring out the limitations of mathematics for exploring the universe, that the deeper one goes the more elusive the quarry becomes. It seems to be like going into a maze—just when we think we are getting near the middle we find ourselves at the exit again. We think we are exploring the universe, but we are really only exploring our own minds—a tiny finite reflection of the Infinite Mind. I don't know if that is what the book is trying to convey, but that is what I get from it. Certainly it underlines the inadequacy of the naïve materialism of the nineteenth century."

This is a remarkable book—a lively adventure story—and yet the title of the book creates a yearning, as my friend so frankly shows, for something deeper. Perhaps a sequel may follow—the mathematician and the imagination.

H. W. TURNBULL.

Les Fonctions définies-positives et les Fonctions complètement monotones. By KY FAN. Pp. 48. 400 fr. 1950. (Mémorial des Sciences Mathématiques, Fasc. CXIV. Gauthier-Villars, Paris)

Positive definite functions of a real variable are those which can be expressed as Fourier-Stieltjes integrals with a monotonically increasing integrand, and completely monotonic functions are those which can be expressed as a Laplace-Stieltjes integral with such an integrand. Alternatively they may be defined as functions satisfying certain inequalities; the identity of the definitions constitutes the fundamental theorems of the subjects, due

respectively to Bochner and Bernstein. Positive definite functions have numerous applications in pure and in applied mathematics, and particularly in the theory of probability, where they form the correlation functions of stationary stochastic processes. Though not as widely used, the completely monotonic functions are also found in many applications.

The present monograph is a very good guide to the subject and to its applications. It is well and clearly written, though few—almost no—proofs are given, and excellently produced. A complete and up-to-date bibliography is given. J. L. B. COOPER.

Formulaire pour le calcul symbolique. Par N. W. MCLACHLAN et P. HUMBERT. 2nd edition. Pp. 65. 350 fr. 1950.

Supplément au formulaire pour le calcul symbolique. Par N. W. MCLACHLAN, P. HUMBERT et L. POLI. Pp. 62. 450 fr. 1950. Mémorial des sciences mathématiques, 100, 113. (Gauthier-Villars)

The contents of the first of these booklets were compiled before the war, and the book appeared in 1941. It has been much in demand as the largest available collection of Laplace transforms. But the years since its first publication have added many new forms to the Laplace dictionary, and these, to the number of some 400, are now printed in the new supplement. This also contains definitions and abbreviations for the functions used in the supplement, an account of general theorems and methods, similar to that in the *Formulaire* itself, but revised and re-arranged, and a couple of pages of corrections to the *Formulaire*, not an excessive amount in view of the large number of complicated formulae contained therein.

The two volumes now give over 1100 transforms, and should be of great value to the ever-increasing number of users of the fashionable Laplace calculus. T. A. A. B.

Œuvres mathématiques d'Évariste Galois. 2nd edition. Pp. x, 61. **Évariste Galois et la théorie des équations algébriques.** Par G. Verriest. 2nd edition. Pp. 56. One volume, 300 fr. 1951. (Gauthier-Villars)

Of the mathematicians universally recognised as great, Galois has left the smallest amount of printed work. Sixty pages contain the researches of this "marvellous boy", killed in a duel at the age of 20. Yet these sixty pages answered age-old problems and set large tracts of mathematics on a new footing. The great problems posed by Abel, "To find all the equations of any given degree which are solvable algebraically", and "To determine whether a given equation is or is not solvable algebraically", were completely solved by Galois. This alone would immortalise his name. What he would have accomplished had he lived beyond youth, it is fruitless to conjecture. For instance, from his mathematical testament, hurriedly scribbled in the closing hours of his life, it seems clear that he had made great strides in the theory of Abelian integrals, but the memoir on this topic to which he refers has never been found.

To a reprint of the 1897 edition of Galois' works, with an introduction by Picard, Gauthier-Villars have added in this volume a reprint of Professor Verriest's essay on Galois and the theory of equations, first published in 1934. The life of Galois is briefly but poignantly sketched, and then the author explains the importance of Galois' ideas in simple language. He restricts himself, properly, to an account of Galois' own work, leaving us to look elsewhere for later developments. This enables the reader to form a fairly precise notion of the extent of Galois' contribution to the subject. No single volume would suffice to describe the influence of that contribution on the subsequent development of mathematics. T. A. A. B.

Moderne Algebra. I. By B. L. VAN DER WAERDEN. 3rd edition. Pp. viii, 292. DM. 24; geb. DM. 27. Grundlehren der mathematischen Wissenschaften, 33. (Springer, Berlin)

This is volume I of the third German edition of van der Waerden's famous treatise (first edition, 1931; second edition, 1937). The volume is by itself a complete textbook on modern higher algebra except for the theory of determinants and matrices; it contains the following chapters: 1. Numbers and sets; 2. Groups; 3. Rings and fields; 4. Polynomials; 5. Field theory; 6. Groups continued; 7. Galois theory; 8. Infinite field extensions; 9. Real fields; 10. Fields with valuations.

The first three chapters are fundamental for what follows, and their contents are developed *ab initio*. The remaining chapters have been arranged to be, as far as possible, independent of each other, and at the end of the table of contents there is a diagram in the form of a genealogical table which shows their logical order. This is most useful for study and reference.

There are two substantial differences between this edition and the last: transfinite induction and well-ordered sets have been re-introduced into Chapter I (they were included in the first edition but omitted from the second), and they are used in the development of the theory of fields, thus making the treatment more general. The chapter on valuation-theory has been considerably lengthened, both by the introduction of new material and by giving more details in the explanations, in recognition of the importance which this subject has lately acquired.

The purpose of the work is to introduce to the reader the ideas, methods and subject-matter of modern abstract algebra. This purpose is excellently carried out. The book succeeds not only in giving a lucid exposition of each particular subject or result under consideration, but succeeds also in the much more difficult task of giving orientation and continuity to the subject as a whole. The reader is given not only an exposition of each topic as it comes, but also an explanation of how it arises, why it is introduced and what purpose it will serve. It is this feature which distinguishes the book from several other works on modern algebra and makes it an excellent text for beginners as well as a useful work for reference.

An improvement could perhaps be made in one respect. The early chapters, although they are complete, are somewhat brief and condensed, references being given to more detailed expositions in other books. It would not take much more space to expand these chapters and make the explanations more detailed, and so help the beginner to understand the subject-matter and its significance more easily. This would be an improvement, and would make reference to other books unnecessary without any considerable increase in length.

But this slight criticism is in no way intended to reflect on the excellence of the book as a whole. At the time when it first appeared (in 1931) this was one of the very few comprehensive books on modern algebra, and it has remained the leading work on the subject ever since. During this time it has played an important part in the teaching of algebra, and so contributed effectively to the development of the subject itself. G. A. D.

The Location of Critical Points of Analytic and Harmonic functions. By J. L. WALSH. Pp. viii, 384. $6. 1950. American Mathematical Society Colloquium Publications, 34. (American Mathematical Society, New York)

The problem with which this book deals is that of determining the position of the zeroes of the derivative of $f(z)$, when those of the analytic function $f(z)$ are given. $f(z)$ is supposed to belong to a certain class of functions, and the aim of the author is to establish the existence of certain regions of the z-plane

which contain all, some or none (according to the particular problem considered) of the zeroes of $f'(z)$. The smaller the class of functions to which $f(z)$ belongs, the more accurately can the positions of the zeroes of $f'(z)$ be described, and the author shows that by imposing on the function $f(z)$ the condition that its zeroes shall possess symmetries of various kinds, elegant and concise results can be obtained.

The first three chapters deal with polynomials. The development is restricted to results and methods needed later, as this branch of the subject has been extensively covered by M. Marden in *The Geometry of the Zeroes*, American Mathematical Society Survey, No. 3. Chapter I is introductory, and includes results fundamental in the subject, Gauss's reduction of the problem to that of finding equilibrium points of a field of force, and the theorems of Lucas, Jensen and Walsh. In the second and third chapters these results are extended and amplified by imposing some condition on the polynomial, (such as reality). Chapters IV and V develop the theory for rational functions, and contain much useful and interesting information. Chapter VI is devoted to analytic functions and the remaining three chapters to the study of harmonic functions.

The technique involved is, on the whole, elementary and the book can be read without an extensive background of mathematical knowledge. It contains many results which have either not appeared before or are not easily accessible. In many cases the methods can be extended, and the book should serve to stimulate research in this branch of mathematics.

H. G. EGGLESTON.

Ortskurvengeometrie in der komplexen Zahlenebene. By W. MICHAEL. Pp. 93. Geb., Sw. fr. 11.50. 1950. (Birkhäuser, Basel)

Certain equations that occur in the mathematics of electrical machines are included in the general form

$$z=(A+Bv+Cv^2+\ldots)/(D+Ev+Fv^2+\ldots), \quad \ldots\ldots\ldots\ldots(1)$$

where A, B, ..., D, E, ... are complex constants, v is a real variable, and so z is a complex variable. As v varies from $-\infty$ to $+\infty$, the locus of the point z in the complex plane, regarded usually in applications as the locus of the end of the vector that joins the origin to the point z, is a closed curve called an "ortskurve" by the author and other continental writers, or "locus-diagram" by English electrical engineers.

By putting $z=x+iy$ and separating real and imaginary parts, the coordinates x, y of a point on the curve can be expressed as rational functions of the parameter v; but the purpose of the book is to study the curves defined by equation (1) directly from that equation, keeping in view as far as possible the vectorial meanings of the complex quantities involved, as these meanings will be prominent in the applications to electrical machines. Special attention is given to these applications, the chief of which are concerned with the following curves:

the circle: $z=(A+Bv)/(C+Dv)$, ...(2)

the circular cubic: $z=(A+Bv+Cv^2)/(D+Ev)$, ...(3)

the bicircular quartic: $z=(A+Bv+Cv^2)/(D+Ev+Fv^2)$. .....(4)

The straight line, to which (2) reduces when C and D are real, and the conic sections, to which (3) and (4) reduce when D, E, F are real, are also fully discussed. The treatment is elementary and detailed, and can be read by anyone with mathematical aptitude and some knowledge of analytic geometry and complex numbers. Several numerical examples of circular cubics and bi-

circular quartics are worked out in the text, at least four of them with numerical values of the constants appropriate to electrical machines.

The electrical engineer will no doubt be glad to have a convenient systematic account of the geometry of these curves, expressed in a notation to which he will at once be able to attach physical significance. F. B.

Paradoxes of the Infinite. By BERNARD BOLZANO; translated with an introduction, by D. A. STEELE. Pp. 189. 21s. 1950. (Routledge and Kegan Paul)

This book is one of a series whose aim is " to present works which have so far been unduly neglected and whose resuscitation is sure to make a real contribution both to the knowledge of the history of philosophy and science, and to the discussion of the great subjects which hold the interest of the world of learning today ". The historical importance of *Paradoxes of the Infinite* well justifies its inclusion in the series; for Bolzano lived when the free and easy outlook of Euler's generation was giving way to the more rigorous standards of Cantor's; and he helped to bring the change about. Although he wrote in many fields—mathematics, logic, philosophy, the physical and social sciences—his work did not have the influence it deserved because of the academic isolation in which he was forced to work.

In this book, first published posthumously in 1851, Bolzano sought to clarify paradoxes concerning infinite aggregates in pure and applied mathematics. The early parts of the book are devoted to the theory of sets. Holomerism (one-one correspondence of the members of a set with the members of a proper subset) is recognised as a property distinguishing infinite sets from finite ones; but the polished concept of the cardinal number of a set was only achieved by later writers. Logical errors in the contemporary use of infinites and infinitesimals are then surveyed and refuted. The book ends with a metaphysical discussion of infinite aggregates and the physical world, employing, among other concepts, a very nineteenth-century ether. This section of the book is certainly the least satisfying to the modern reader.

The translation occupies little more than half the present book. The rest contains an excellent introduction and commentary, a full bibliography, a key to all the quotations and an index. Dr. Steele helps us to appreciate the intellectual climate in which Bolzano worked, and his praise is always tempered by readiness to admit aspects in which Bolzano's writings disappoint. We can in fact be grateful to Dr. Steele for providing not only a very readable translation, but also an attractive piece of original scholarship. T. J. F.

Space-Time Structure. By E. SCHRÖDINGER. Pp. viii, 119. 12s. 6d. 1950. (Cambridge University Press)

This is a book that can be read with profit and pleasure by anyone possessing a rudimentary knowledge of general relativity theory. With profit, because probably nowhere else will be found the mathematics of the subject so well arranged from the standpoint of showing how much has to be assumed about space-time in order to establish any particular result. With pleasure, because the exposition is given by an absolute master of the craft who succeeds in imparting his own vivid insight with a sprightly inventiveness of idiom in his adopted tongue.

All but the last of the dozen short chapters are devoted to what could be regarded as a development of the fundamental parts of general relativity theory using orthodox mathematical methods. It is the *sequence* in Professor Schrödinger's development that is unusual and significant.

His object is to " investigate the geometry of our [space-time] continuum in three steps or stages, viz.

(1) when only general invariance is imposed ;
(2) when in addition an affine connexion is imposed ;
(3) when this is specialized to carry a metric ".

His special concern is to show at each stage how various notions receive a meaning, which they could not have been given at an earlier stage, but which does not have to await a later stage.

Part I entitled " The unconnected manifold " treats of tensor algebra, tensor densities, and derivatives in so far as tensors, such as a *curl*, can be constructed from ordinary derivatives. Illustrating this work, the author shows that it suffices to express Maxwell's equations of the general electromagnetic field in invariant form, though not the accompanying " material equations " corresponding to the usual relations between the vector-pairs (**H**, **D**) on the one hand and (**B**, **E**) on the other.

Part II is on the " Affinely connected manifold ". The author shows how the introduction of an affine connexion leads to the definition of invariant derivatives. He then introduces the notion of parallel transport and shows how it gives an alternative way of defining an affine connexion. Proceeding to what is perhaps a somewhat intuitive investigation of the condition for the integrability of an affine connexion, he defines the Riemann Christoffel tensor. As a further application of the notion of parallel transport he then defines the geodesics of an affine connexion. It is another illustration of Professor Schrödinger's development that he here shows, before introducing a metrical tensor, how an affine connexion makes possible the comparison of lengths along a geodesic, but not between different geodesics. He concludes this part with a statement of the aims of general relativity regarding its treatment of gravitation, and shows that these cannot be fulfilled without some restriction upon the geometry beyond what is provided by imposing only an affine connexion.

Part III is on the " Metrically connected manifold ". This means, of course, the passage to Riemannian geometry. But, as soon as he has defined this, the problem to which Professor Schrödinger addresses himself more fully than is usually done is that of the necessary and sufficient conditions for an affine connexion to be compatible with the existence of the metrical tensor of Riemannian geometry. Compatibility here means that the latter should give the same result for distance along a geodesic as the affine measure referred to above. He then goes carefully into the physical interpretation of the metric when the affinely and metrically connected four-dimensional continuum is taken to provide a model of the physical world. He then proceeds to formulate the conservation laws of general relativity and Einstein's variational principle resulting in Einstein's field-equations.

It is hoped that this brief review of Professor Schrödinger's presentation will convey the fact that it is designed to show where opportunities of generalization are to be sought. The purpose of such generalization is, of course, to obtain a " unified field-theory " for gravitation and electromagnetism. The book shows, with a considerable approach to completeness, the successive restrictions that are imposed upon the continuum in order to obtain the gravitational field-equations. It is to be expected that the required generalization is to be found by dropping certain of the later of these restrictions. It is also to be expected that the generalized field-equations themselves will be derivable from a variational principle on the general model of that used by Einstein. This is the point of view adopted by Professor Schrödinger in his final chapter.

He recalls the method of Palatini of treating the variational principle in which the components of the affine connexion and of the metrical tensor are regarded as independent functions for the purpose of the variation. The

resulting " field-equations " then include the required relations between the connexion and the metric. He shows how this method can be extended to the case where the connexion and the metrical tensor are non-symmetric. The result is what he calls the Einstein-Strauss theory, and is very similar to Einstein's most recent version in *The Meaning of Relativity* (1950 edition, reviewed in *Math. Gazette*, **35** (1951), pp. 127-9). Professor Schrödinger finally presents a sketch of his own " purely affine " theory, which combines certain features of the Einstein-Strauss theory and of Eddington's affine theory.

It appears that Professor Schrödinger's book was in the press before Einstein's was published. It is therefore remarkable that both these eminent thinkers conclude by expressing the conviction that the theories obtained in the manner described are the most *natural* extensions of the theory of general relativity—even though it seems that neither has reached any detailed conclusions regarding the physical interpretation of the results.

W. H. McCrea.

Verstandliche Elemente der Wellenmechanik. I. By K. Jellinek. Pp. xii, 304. 34 Swiss Fr. 1950. (Wepf & Co. Verlag, Basel)

This book, the first of two volumes, is a fresh attempt to describe to experimental students, in terms they are likely to understand, the fundamental ideas of the newer physical theories. In substance it follows a familiar path. There is first a longish section (about 90 pp.) on " Wave Mechanics and Light ", in which the wave and corpuscular theories are discussed and then reconciled through the Heisenberg uncertainty principle. The next section repeats this more briefly for material particles. A few short sections describing some essential features of the solutions of differential equations, notably the wave equation, lead to Schrödinger's equation, and the stage is then set for a more detailed discussion of the simpler one particle problems : the harmonic oscillator (in one and three dimensions), the rigid rotator, and the hydrogen atom. Finally the author considers a particle moving in a potential well, including the tunnel effect and its application to alpha-particle emission.

There is, however, very much to commend in Dr. Jellinek's treatment. His style of writing is concise and clear. He is concerned throughout to encourage the reader's physical understanding of the subject, and to this end rightly prefers the wave-mechanical approach of de Broglie and Schrödinger to the more abstract ideas of Dirac and Heisenberg. He is also at pains to compare the results of classical and wave mechanics, and in several places gives in turn the classical theory, the Bohr theory, and the final wave-mechanical treatment of the problem. One feels that this should appeal particularly to students well-grounded in classical mechanics, and able to appreciate the underlying continuity in theoretical development, as well as the new ideas. Another attractive feature is the unusually large number of excellent diagrams (though here and there the drawing leaves something to be desired), which contribute much to the physical picture of what is happening.

A slight reservation may be made about the first section. Here again the description of interference, beats, phase and group velocity, and the propagation of wave packets, is admirable. About 30 pages are then inserted on relativity theory, including the Lorentz transformation, the mass energy relation, the space time continuum, and even a brief sketch of some (needless to say, not *all*) past and current cosmologies. The reader may feel with the reviewer that too much precious space is used in justifying the concept of a universal medium (" Weltäther ") for light waves, though this is clearly a subject dear to the author's heart. This first section, indeed, covers so much ground that many students, though their interest may have been aroused, will

retain on many points only a vague impression. References for further reading are, however, fairly abundant.

Nevertheless, this is a book to be recommended, even to students whose knowledge of German barely extends to the borders of " Basic ". The type is clear and readable. R. A. B.

An Introduction to the Calculus of Variations. By CHARLES FOX. Pp. viii, 271. 21s. 1950. (Geoffrey Cumberlege, Oxford University Press)

On the flyleaf of this book it is claimed that " In this work the Calculus of Variations is developed both for its intrinsic interest and because of its wide and powerful applications to modern Mathematical Physics." It appears, however, that the author's main preoccupation is with the applications and that his interest in the Calculus of Variations derives from its applicability to physical problems rather than its intrinsic discipline. Thus, for example, Hamilton's principle is proved, from dynamical considerations, in two different ways for holonomic systems and again in two different ways for non-holonomic systems; and excellent accounts are given of the subject of elasticity and of the equilibrium conditions of a continuous elastic medium, with full definitions, before any discussion is attempted of the Saint-Venant torsion problem. On the other hand, the purely mathematical parts of the book are not always treated with such comprehensive care. The phrase " simply-connected region " is used on p. 238 without (to the reviewer's knowledge) any prior introduction. In § 5.22, it is shown that the Hamiltonian of a function L, $=L(q_1, \ldots, q_i, \ldots, q_n)$, which is homogeneous of degree one in the q_i $(i=1, \ldots, n)$, is zero. A converse is stated, namely, that if, between the variables q_i, p_i $(=dL/d\dot{q}_i)$, there exists a relationship given by $K=K(q_1, \ldots, q_n, p_1, \ldots, p_n)=0$, then K is the Hamiltonian of L. It is difficult to see how more can be claimed than that the Hamiltonian of L divides K. A more pedantic suggestion, perhaps, is that the words " the truth of Theorem 6 " on p. 42 be replaced by " that the conclusions of Theorem 6 follow ".

The book provides an excellent introduction to the subject for those primarily concerned with having available techniques and rules of procedure for tackling concrete problems, and a most exhaustive treatment of the classical problems of the Calculus of Variations (*e.g.* the brachistochrone, the surface of minimum resistance, the principle of least action) is among the best features of the book. However, for the student of pure mathematics, it does not seem to replace Hadamard's *Leçons sur le Calcul des Variations* or Bliss's *Calculus of Variations* (whose mathematical content broadly coincides with its own).

There are two strange omissions in connection with the solutions of the accessory equation, (i) the elementary proof that the definition of conjugate points is independent of the choice of solution, and (ii) (in an otherwise most satisfactory summary at the end of Ch. 2) a statement of the conditions under which $u_1(x)/u_2(x)$ increases or decreases as x increases, if $u_1(x)$, $u_2(x)$ are independent solutions. Some troublesome misprints also occur (*e.g.* (v) of Theorem 16). However, these last are not serious detractions from the book's merit and suitability to the applied mathematician or theoretical physicist.

P. H.

The Mathematical Theory of Plasticity. By R. HILL. Pp. ix, 356. 35s. 1950. (Geoffrey Cumberlege, Oxford University Press)

The English translation of Dr. Nádai's German book on *Plasticity* for many years has been a valuable textbook for those who need an introduction to the subject. The theory of plasticity is, however, a comparatively young science,

and since Dr. Nádai's book was written many changes have taken place in the theories which form the fundamental basis of plasticity. Many of these changes have occurred in recent years, and the need for a new book (in English) has become a matter of urgency. The book by Dr. Hill is very welcome, and is thoroughly recommended to engineers, physicists and applied mathematicians who need a modern systematic introduction to the mathematical theory of plasticity. As far as possible the theory is developed with mathematical rigour, but the mathematical knowledge which is needed for an understanding of the theory should be within the capacity of engineers and physicists. The physical background of the theory is carefully discussed, and the expression of physical ideas in analytical form is often helped by the use of geometrical diagrams. The treatment of the various subjects which are considered is taken to the limit of present knowledge (1949), and includes some important contributions which have been due to the author himself.

Chapter 1 contains a brief historical outline of plasticity and an account of the physical background of the theory. This is followed by Chapter 2 on the foundations of the theory, which is valid for an ideal isotropic plastic body. Special consideration is given to the yield criteria of Tresca and von Mises, and the theory of strain-hardening is developed, leading to the equations of Lévy-Mises and Reuss. The complete stress-strain relations of Reuss, in which von Mises' yield criteria is used, form the basis for most applications in the book when strain-hardening is important. If strain-hardening is neglected the yield criteria of either von Mises or Tresca are used. Chapter 2 closes with a brief account of other theories, and in particular mentions that of Hencky, but the author points out that Hencky's equations are unsuitable to describe the complete plastic behaviour of a metal.

Chapter 3 contains general theorems on uniqueness, also extremum and variational principles. Some change of notation seems to have been made in equation (1) of Chapter 3, compared with equation (22) of Chapter 2, which becomes evident in the special case of the Reuss equations, and this is at first reading rather confusing.

Chapters 4 and 5 are concerned with plastic-elastic problems for which analytical solutions are known, either in an explicit form or such that the solutions can be completed by direct numerical integration. The problems include those in which all elements are simultaneously stressed to the yield limit so that no plastic-elastic boundary has to be calculated, and those in which plastic-elastic boundaries exist side by side. Chapter 5 includes an excellent discussion of the autofrettage process.

Chapter 6 contains the theory of plane plastic strain applied to a plastic-rigid material, and the geometry and numerical calculation of slip-line fields is fully discussed. The ideas and techniques of this chapter are applied in the following three chapters to a variety of problems of plane strain, which include the analysis of stress and strain in technological processes such as rolling, forging and drawing. Steady motion problems are discussed in Chapter 7 and non-steady problems in Chapters 8 and 9. The problems of wedge-indentation and yielding of notched bars under tension, the indentation and the theory of hardness tests, are among the non-steady problems which are considered.

Chapter 10 deals with axially symmetrical distributions of strain, and is followed by a chapter on miscellaneous topics such as the process of deep-drawing, the general theory of sheet bending, the plane strain of a general plastic material, and the theory and application of plane plastic stress. The closing chapter deals with plastic anisotropy.

Four appendices are added which will be helpful to engineers and physicists. In Appendix 1 suffix notation and the summation convention, which is the

notation of cartesian tensors and which is used throughout the theoretical part of the book, are explained. Appendix 2 gives the stress equations of equilibrium and the components of the strain-rate tensor in cylindrical and spherical polar coordinates. Characteristics of differential equations are discussed in Appendix 3, and transformation equations for stress components in a plane are given in Apendix 4.

In the reviewer's opinion this book should be in the possession of all those who wish to undertake study and research in the theory of plasticity. The usual high standard of presentation and printing of the Oxford University Press has been maintained. A. E. G.

Mathematical Snapshots. By H. STEINHAUS. New enlarged edition. Pp. vi, 266. 27s. 6d. 1950. (Geoffrey Cumberlege, Oxford University Press; New York)

This new edition of the most charming of all the books of the "mathematics for everybody" type is very welcome. The possessors of the first edition must count themselves fortunate, however, since the new edition is slightly less attractive than the original. The layout is too Americanised for my taste, but this is a criticism of comparatively little importance. More serious regrets are that there is now no colour, the anaglyphs are replaced by photographs of models, the delightful self-unfolding dodecahedron and other gadgets are gone, and in some instances replacements have supplied poorer pictures, as, for example, the shadow-pictures of the conics. Austerity is no doubt largely to blame. Against this, the improvements are substantial: 295 diagrams instead of 180; an increase in the amount of comment; a more systematic and orderly arrangement.

The professional mathematician in Professor Steinhaus, expert in the austere abstractions of the theory of the real variable, must have been highly amused at the antics of his other self, revelling in models, pictures, maps, chessboards, shadows, puzzles and jokes. But if the "sandwich" theorem is a joke, it is serious mathematics as well: "it is always possible to cut a sandwich with a plane stroke so as to halve the bread, the butter and the ham." The young mathematician, attracted by the charm of the book, may well be stimulated by a theorem such as this to pursue his routine studies with an increased zest, and the volume ought therefore to be in every school library. But this ulterior motive, however important, need not be unduly stressed. The book is a magnificent store of visualised ideas, and for this reason alone it is one which mathematical shelves, library or private, should contain. Even at its present-day price, it is well worth the money. T. A. A. B.

Projektive Differentialgeometrie. I. By G. BOL. Pp. xii, 365. DM 19.80; geb. DM 22.50. '1950. Studia Mathematica, 4. (Vandenhoeck & Ruprecht, Göttingen)

This book is the first of two volumes in which the main advances in Projective Differential Geometry during the last fifty years will be treated. This first volume deals mainly with curves and curve-strips (tapes), and the theory of surfaces will appear in the second volume.

The treatment differs from that of other authors in that the normalization of projective coordinates is avoided altogether. Instead the author combines a parameter transformation with a change of factor of proportionality in such a way that a certain determinant remains invariant.

The first chapter is devoted to the study of plane curves, introducing Frenet formulae with two functions a and b. The vanishing of b characterizes a sextactic point. When b does not vanish, it is used to define projective arc, and a is the projective curvature. Curves which admit a one-parameter

group of projective transformations are those with constant curvature, and are called W-curves, first studied by Lie. In this first chapter there is studied the possibility of an analogue in projective geometry of the property of a straight line in metric geometry of being an extremal curve. The corresponding curves in this case are those for which the projective curvature, regarded as a function of the projective arc, satisfies a certain differential equation. There is also in this chapter a projective analogue of the four-vertex theorem in Differential Geometry in the Large.

The second chapter deals with projective geometry in space. It contains some Line Geometry and treats such topics as Lie and Darboux quadrics, Darboux and Segre directions, conjugate nets and Laplace transforms. It ends with a study of contact of curves and surfaces.

The third chapter, dealing with skew curves, has many topics which are treated by the author's own methods, and in particular he has a new and very natural method of introducing the harmonical C3's of Fubini. This chapter and the next one, dealing with curve strips, are based upon a series of original papers published by the author.

The book is self-contained and clearly written, demanding little in the way of previous knowledge. A useful feature is the inclusion, at the end, of topics from other branches of mathematics which are needed for the understanding of this book. They include such topics as Vectors and Projective Geometry, systems of homogeneous linear differential equations, and the Theory of Quadratic Differential Forms. E. T. D.

Introduction to Algebraic Geometry. By W. Gordon Welchman. Pp. viii, 349. 25s. 1950. (Cambridge University Press)

"The aim of the book is to lead a student as rapidly as possible to the study of configurations, loci and transformations in space of three, four and five dimensions." The contents of the book, which is admirably produced in the best Cambridge Press style, are : .

Chapter I. Introduction and definitions. Chapter II. Algebraic systems and correspondences. Chapter III. Rational systems of freedom one. Chapter IV. The conic. Chapter V. Configurations. Chapter VI. Metrical geometry. Chapter VII. Homographic ranges on a conic. Chapter VIII. Two conics, reciprocation of one conic into another. Particular cases. Chapter IX. Two conics. Apolarity. Chapter X. Two-two correspondences. Chapter XI. Applications of matrix algebra. Chapter XII. Invariants and covariants.

The body of the book deals, therefore, with conics. The background for higher study is mostly provided by the first three chapters. There were two courses open to the author when he wrote these three chapters. He could have proved all his results as he went along, assuming the minimum of algebraic knowledge in his readers. This is the current fashion. Or, by means of adequate references and explanations, he could give the reader a fair idea of the theorems he would wish the reader to use in the course of the book. Dr. Welchman has chosen the second course.

It must be pointed out that parts of the first three chapters could easily have been improved. In 1.44, for example, the *incidence theorem* is discussed. There is a clear explanation of this important theorem, which connects the dimension i of the intersection and the dimension j of the join of two linear spaces S_r and S_s in S_n, but the theorem itself,

$$i+j=r+s,$$

is not proved. We are told that it *can* be proved by using the theory of

linear homogeneous equations, to which references are given. It is a great pity that the proof, which is extremely simple, is not given.

The style of the book is excellent, as anyone who heard the author lecture in Cambridge some years ago would expect. It is unfortunate that the war prevented the publication of the book when it was first written, ten years ago. In the meantime J. A. Todd's *Projective and Analytical Geometry* (Pitmans, 1947) has appeared, and Todd's book not only goes much further, dealing with quadrics and twisted cubics, but it also has a large selection of examples. Welchman's book contains none. For the more mathematically-inclined reader I would suggest Todd's book, but since there are certainly many who find the pace in Todd too hot, there will certainly be a good market for the book under review. D. PEDOE.

Geometry. By H. G. FORDER. Pp. 200. 7s. 6d. 1950. (Hutchinson's University Library

According to the dust-cover this book is " An expository book for the non-specialist who is interested in geometrical shapes. Beginning with elementary geometry, treated in a fresh way, it includes discussions on projective, algebraic, non-Euclidean and differential geometry, the general geometry of higher dimensions, and the logical structure of geometries."

The promise held out on the cover is amply fulfilled in this book, which is an amazing *tour-de-force*. In his preface the author, who is a very distinguished geometer, says : " . . . it is hoped that the greater part of most chapters will be readily understood. If towards the end of any chapter, the reader finds it too strenuous, he should turn to the next for a fresh beginning." It seems to the reviewer that only the very seasoned geometer will be able to read through this book without accepting this advice. He will find much to ponder on in each chapter. For the amateur, this book can be recommended for the excellent survey it will give him of practically the whole field of geometry.
D. PEDOE.

Algebraic Curves. By ROBERT J. WALKER. Pp. x, 201. 25s. 1950. (Princeton University Press ; London : Geoffrey Cumberlege)

This book provides an introduction to the new methods in algebraic geometry which have been developed during the past twenty-five years. A reader in search of the material in this book would have to read treatises in German and Italian, some of them out of print at the present moment. Here is a book, in which the key-note is rigour (and it should be said at once that American " rigor " is as good as English " rigour "), which takes us as far as the Riemann-Roch theorem for algebraic curves, the culmination of the classical theory.

The author devotes the first chapter to algebraic preliminaries, and the second to the theory of projective spaces, so that the book is practically self-contained. Even the remainder theorem is proved, and it is pleasing to see that the traditional proof is given. No doubt the author was not fortunate enough to see a recent fulmination in the *Gazette* against this proof, but the eminent algebraist who, according to the preface, made a very careful examination of the work before it was printed must be held equally responsible if, indeed, the traditional proof is incorrect. Undeterred also by André Weil's refusal to have anything to do with elimination theory, Professor Walker bases a large part of his work on the theory of the resultant of two forms, and the reviewer found this very carefully and pleasingly done.

Chapter III deals with plane algebraic curves. The very careful proof that " if two cubics intersect in exactly nine points every cubic on eight of these points is on the ninth " will be noted. The delightful consequences drawn

by Salmon in the classical *Higher Plane Curves* will be remembered. Most of them are given in Walker's book, but not as immediate deductions. In a recent address to the International Congress of Mathematicians at Harvard, Professor Zariski, talking about modern methods in algebraic geometry, said: "Our aim is not to prove that our fathers were wrong, but that they were right!" One remembers this constantly while reading Walker's book.

In the section headed "Analysis of Singularities" on page 74 there is a slip. We are told that the curve

$$x^4 + x^2y^2 - y^2 - 2a^2x^2 + a^4 = 0,$$

that is

$$(x^2 - a^2)^2 - y^2(1 - x^2) = 0$$

is irreducible for any value of a, and for $a \neq 0$ it has ordinary double points at $(\pm a, 0)$ and one at the infinite point on the y-axis. The second statement is only true if $a^2 < 1$, and if $a^2 = 1$ the curve is evidently reducible. The argument, which introduces "infinitely near" double points, is not affected. It is proved that any irreducible curve can be transformed into one having ordinary singularities by a succession of quadratic transformations. Here, again, is a theorem which has tripped up even the most eminent in the past.

Chapter IV is headed "Formal Power Series", and develops the idea of a "place" on an irreducible algebraic curve. This is done formally, over a general ground field, and is most interesting. It is also rather difficult, and perhaps not suitable for the beginner. There is a slip at the top of page 89, where the order of f is defined to be the integer k. It should be $-k$. A place of a curve C having been defined as an equivalence class of irreducible parametrisations, the author is able to deal with the problems raised by the intersections of two algebraic plane curves, to prove Bezout's theorem, and to establish the Pluecker formulae. This chapter ends with a proof of Noether's $Af + Bg$ theorem, and the interesting applications which may now be made to the intersections of two cubic curves.

Chapter V, "Transformations of Curves", deals very thoroughly with the rational and birational transformations of an algebraic curve. All the necessary theorems on algebraic extensions of fields are proved. Space curves are defined, projection as a rational transformation considered, and many theorems which are often lightly skipped over are proved. To add to the richness of this chapter there is a section on valuations which deals very adequately, once again, with matters too lightly assumed in other texts.

Having the ground well-prepared, Chapter VI on "Linear Series" produces results very quickly. In thirty pages the theory is developed as far as the Weierstrass Gap Theorem "For any given place P of a curve of genus p there exist precisely p values of n, all $\leqslant 2p - 2$, for which nP is not the set of poles of a rational function". The final section in this very attractive chapter deals with the geometry on a non-singular cubic. Here, almost on the last page, will be found Salmon's theorem on the four tangents to a non-singular cubic from any one of its points.

There is an excellent index, and exercises are given after many of the sections. It is a pleasure to read a book which so excellently combines the charm of the old algebraic geometry with the rigour of the new.

D. PEDOE.

The Theory of Algebraic Numbers (Carus Mathematical Monograph Number Nine). By HARRY POLLARD. Pp. 143. 24s. 1950. (John Wiley, New York: Chapman and Hall)

This is an elementary introduction to the classical theory of algebraic fields. The first two chapters discuss factorisation for rational and Gaussian integers; an example shows that unique factorisation fails in $R(\sqrt{-5})$. The third

chapter is devoted to algebraic preliminaries. The remaining eight chapters develop the theory of general algebraic number-fields. They culminate in proofs of the uniqueness theorem for factorisation of ideals, of the finiteness of the class-number, of the Dirichlet-Minkowski theorem on the existence of units and of the fact that all primes which ramify divide the discriminant (but not the converse). As an application it is shown that $x^p + y^p = z^p$ has no integral solution with $p \nmid xyz$ if the prime $p \neq 2$ is regular.

The exposition is on familiar lines, but very readable and well done. Only a school knowledge of mathematics is assumed. Everything else required is proved, except that two results are clearly enunciated with reference to proofs.

The general layout of the printed page is clear and pleasant to the eye; and the text is laudably free from misprints. There is an index.

In every way this is a book to be recommended to the beginner with no particular algebraic equipment. It could be read with advantage by an intelligent sixth-former. J. W. S. CASSELS.

The Arithmetic Theory of Quadratic Forms (Carus Mathematical Monograph Number Ten). By BURTON W. JONES. Pp. x, 212. 24s. 1950. (John Wiley, New York: Chapman and Hall)

There is a lack of modern books on quadratic forms in many variables. The two-volume work of Bachmann (1898, 1923) is still good reading but now out of date, while Dickson's *Studies* (1930) is confined to a few topics and more old-fashioned in outlook than Bachmann. We need an introduction on modern lines in which, for example, a genus is defined by equivalence to an arbitrary modulus (as was done already by Minkowski), not by a heterogeneous assemblage of "characters" (as in, say, Dickson). We also need a systematic account of the great advances of the last thirty years. The book under review, which attempts to fill both these requirements in just over 200 octavo pages, is a valuable addition to the literature but, not unnaturally, fails to be completely satisfactory in either rôle.

The book is intended for the beginner. Only elementary matrix theory is assumed; the theory of p-adic numbers (which is used from the start) is developed *ab ovo*. It is however, possibly, too concentrated for such readers. In Chapter VII, for example, the classical theory of binary forms is crammed into 46 pages, together with the necessary parts of the theories of Pell's equation, of ideals in quadratic fields and of continued fractions. Further, some essentials in an introduction are lacking: the finiteness of the class-number of forms in n variables with given non-zero determinant is tacitly assumed, but does not seem to be explicitly stated, far less proved; and there is no discussion of integer forms of zero determinant.

The more knowledgeable reader, on the other hand, will find a useful connected account of those recent fields of activity of particular interest to the author. Some of the results and proofs are new. The first three chapters develop systematically the theory of forms in fields, particularly the rational and p-adic fields, and lead up to Hasse's theorem that equivalence in all p-adic fields implies rational equivalence. The next two chapters deal with forms in p-adic and rational integers. They contain a generalisation of Witt's lemma to p-adic integers and Siegel's theorem that equivalence in all p-adic integer rings implies "semi-equivalence". Chapter VI opens with a brief account of Siegel's formula for the number of representations by a genus and of recent work on the representations by individual classes (including an unpublished Cornell thesis). Only results are quoted and there is no mention of the suggestive work of Linnik or Hecke. The rest of the chapter and Chapter VIII contain a detailed account of the author's theory of integral

representation of forms or numbers by forms which is different in shape, but not apparently in essence, from the classical theory of Gauss and Minkowski.

There are a few slips. On page 189 we read, " It is not known whether or not every genus of indefinite [ternary] forms has but one class," whereas genera of more than one class were extensively studied by Meyer [*Crelle's Journal*, 98 (1885), to 116 (1896)]. The proof of lemma 2 appears to be incomplete. Theorem 38a requires $p = 2$.

This is perhaps a difficult book to read. Proofs in the theory of quadratic forms often require consideration of a host of details and special cases, but the detail here is sometimes badly managed or omitted. The use of the term " congruence " instead of " equivalence " in a field is an unfortunate innovation, particularly as congruences in the usual sense also occur.

There is a bare bibliography of works quoted but no suggestions for further reading (*e.g.* Bachmann, Minkowski, Witt are absent), a useful index of theorems as well as an ordinary index, and a collection of 95 problems.

J. W. S. CASSELS.

Contributions to Mathematical Statistics. By R. A. FISHER. $11\frac{1}{4}'' \times 8\frac{1}{2}''$. About 660 pages. 60s. 1950. (John Wiley, New York ; Chapman and Hall, London)

During the half-century which has just closed, Professor R. A. Fisher has influenced the theory of statistics more than any other person now living. This is not only because of his own contributions, but also of the much greater aggregate of work which can trace its inspiration to Fisher's writings. Even his opponents got started on something he wrote. To the average student of statistics many of Fisher's writings have been rather inaccessible for one reason or another, although some photographic reprinting has been done before now in America. The present volume consists of a re-issue of 43 papers selected by the author as most likely to be of service to others. They are photographically reproduced, on a page large enough to contain extracts from the *Philosophical Transactions of the Royal Society*, and therefore leaving lots of room for marginal notations by the reader in the case of many papers from the smaller journals. The Editor is Walter A. Shewart, general editor of the Wiley publications in Statistics, and evidently a hero-worshipper, since he claims that the method of reproduction preserves for posterity the flavour of the original papers as well as their content. Fortunately, this aim coincided with a laudable desire to reduce the cost of publication as much as possible. At 12 shillings per pound of weight, the book may be considered good value, but nevertheless the overall price is likely to be a considerable deterrent to its extensive sale in this country.

Professor Fisher contributes a preface, and also introductory notes to most of the papers, which form interesting, if at times provocative, reading. A biography of Fisher by Professor Mahalanobis is reproduced from an Indian journal, and the book concludes with an exhaustive subject-matter index which has been specially compiled by Professor John Tukey. The original page-numbering has been preserved as the decimal part of a number in which the numbers, in Clarendon type, before the decimal refer to the serial order of the papers. The opportunity has been taken to make many corrections, some of which have led to whole pages of the present volume being crossed out, being replaced by specially printed addenda. There is a prefatory note to Paper 1, but the paper itself is missing, since it could not be reproduced.

These are unusual features in a book, which will nevertheless serve a useful purpose in rendering many of Fisher's papers more accessible to the student. No attempt will be made here to assess the value of Fisher's thirty years of

writings. The papers may be left to speak for themselves, aided by the prefatory notes. To the reviewer there is only one discordant note. It may surprise some to learn from the author's note to Paper 29 that this is the only paper involving personal criticism he had had reason to write. However that may be, it is open to question whether a paper consisting of a personal attack on Professor Karl Pearson soon after the latter's death should have been selected for reproduction as likely to be of service, and a matter for regret that the attack should have been renewed, with considerable venom, in the author's note written during the preparation of this volume. J. WISHART.

An Introduction to Probability Theory and its Applications. I. By WILLIAM FELLER. Pp. xii, 419. 48s. 1950. (Wiley, New York; Chapman and Hall)

Dr. Feller, Professor of Mathematics at Cornell University, is already well known in this country for his contributions to the mathematical theory of probability and for his special interest in stochastic processes. In this book his explicitly stated aim is to present probability theory " as a self-contained mathematical subject rigorously, avoiding nonmathematical concepts ", making a " serious attempt to unify methods ". The book is not, however, intended to be purely formal. Wherever possible the author indicates the practical applications of his theoretical results to problems in many fields of science and engineering. In this volume, the first of two, the author succeeds admirably in his task. He never loses sight of his primary aim, the lucid development of the mathematical theory; and though the descriptions of applications are adequate, informative and entertaining, they are never allowed to obscure the main argument. The reader always knows precisely where he stands and whither he is being led.

Following von Mises the author begins by relating probabilities to a *Merkmalraum* or *sample space* in which every conceivable outcome of an experiment is, by definition, completely described by one, and only one, sample point; an *event* is then defined as an aggregate of sample points. With each sample point it is assumed that there is associated a non-negative number, called its *probability*, such that the sum of the probabilities of all points in the sample space is unity. The probability of an event is then defined as the sum of the probabilities of all sample points in it. With this avoidance of metaphysics the foundations of an axiomatic treatment are laid. In the development of the theory the author acknowledges his debt to Kolmogorov, though, as in this volume sample spaces are restricted to those which are discrete, measure theory is not much in evidence.

The first chapters give an elementary introduction to problems of sampling with and without replacement. The Maxwell-Boltzmann, Bose-Einstein, and Fermi-Dirac " statistics " of the physicists are briefly explained, and serve to illustrate the impossibility of creating probability models *a priori*. In Chapters 6 and 7 the Binomial distribution is discussed and the Poisson and Normal approximations to it are derived. The Poisson and Normal distributions are then considered on their own merits with many practical illustrations. The next chapter lies outside the main path of development, but the reader is clearly informed of this fact and is correctly advised that it can be omitted on a first reading as the results established in it will not be required later. The chapter contains a discussion of the fundamental properties of randomness emphasised by von Mises, and concludes with a formal proof of Kolmogorov's *law of the iterated logarithm* for the special case of Bernouilli trials. The main development is continued in Chapter 9 with a discussion of random variables, defined as functions on a sample space, and of expectation; Chebyshev's inequality for a random variable and Kolmogorov's

stronger inequality for n mutually independent random variables are proved. The limit theorems are continued in Chapter 10 with discussion of the central limit theorem and proofs of the laws of large numbers using the method of " truncation ". Chapter 11 treats of sequences of integral non-negative random variables and of their special properties that can be investigated by means of generating functions. The elegance and power of this method are amply demonstrated in the exploration of convolutions and of compound distributions with application to nuclear chain reactions. Practical difficulties of determining explicitly the probabilities from a given generating function are faced, and the method of partial fractions is explained and illustrated.

The remainder of the volume consists of six chapters on recurrent events, random walks, and Markov chains, subjects of great interest and practical importance which one would hardly expect to find in a book that claims to be only an introduction. Yet the author completely justifies the inclusion of these subjects by giving a coherent and interesting account of them without recourse to advanced techniques. The theory of recurrent events is developed in Chapters 12 and 13. The *arc sine law* is proved, and its surprising implications are clearly illustrated by numerical examples of processes depending on the cumulative effects of chance in sequences of Bernoulli trials. The theory neatly disposes of problems concerning runs of successes and self-renewing aggregates. In the next chapter the classical gambling problems and random walks are considered, with applications, as a first approximation, to diffusion and the Brownian motion. The proof is given of Polya's theorem that in one and two dimensions the wandering particle is certain to return in due course to its initial position, whereas in three dimensions there is a positive probability that it will never return. The chapter concludes with a simplified account of sequential sampling for discrete variables. Chapter 15 is a useful and lucid introduction to the theory of Markov chains. It begins with consideration of chains with constant transition probabilities, and concludes with a brief discussion of more general Markov processes. The author bases his analysis on his general theory of recurrent events, a new approach which simplifies and unifies a subject on which information has hitherto been widely scattered. The algebraic treatment of finite Markov chains is completed in the next chapter ; their ergodic properties are established by the use of generating functions and the expansion of partial fractions. The last chapter is a miscellany of results concerning simple stochastic processes. The treatment of these processes is limited by the author's self-imposed restriction to discrete sample spaces, but nevertheless some interesting applications to radio-active transformations, various " birth and death " processes, telephone traffic, and machine maintenance, are briefly discussed.

The examples deserve special praise. There are over 300 of them, and many develop and complement the arguments of the text. For the more difficult problems hints for solution are provided ; answers are given to them all.

Throughout the text numerous references to original sources are given ; back references within the text knit together the main theme of theoretical development. An index is provided.

The author is to be congratulated on making readily available in English an introduction to subjects which have not yet attracted much interest in this country. His publishers and printers have also helped to make this excellent textbook a pleasure to read and refer to. Both teachers and students will find that this book has much to offer them ; it should be read by all who are interested in probability theory and its applications.

B. C. B.

Differential and Integral Calculus. By E. LANDAU. Translated by M. HAUSNER and M. DAVIS. Pp. 366. $4.50. 1950. (Chelsea Publishing Co.)

Fourier Series. By W. W. ROGOSINSKI. Translated by H. COHN and F. STEINHARDT. Pp. vi, 171. $2.50. 1950. (Chelsea Publishing Co.)

The list of reprints and translations being made available by the Chelsea Publishing Company of New York is impressive and attractive. Many of the most important continental treatises of the inter-war period, now unobtainable in their original form, are being put at our disposal. Two of the most recent translations are noticed here, but readers would do well to get a Chelsea list for themselves and see what is available.

Landau's book, first published in 1934, represents that great mathematician's final thoughts on the exposition of the calculus. Written with the typical Landau economy of words, it suffers very little by translation, and is, in my opinion, the best book on calculus for the professional mathematician in the making, for the born analyst in his first year at a university. There are no applications, no examples, no geometrical illustrations ; Landau was not writing for weaklings. But open the book at random : precision, clarity and power are so fully exhibited that surely no one would need to look at the title page in search of the author's name ; *ex ungue leonem*.

It should be added that Landau's superb presentation presupposes a knowledge of the theory of the real number, such as is given, for instance, in his own little book, *Grundlagen der Analysis*, and that a reprint of this is also available in the Chelsea series.

Professor Rogosinski's booklet on Fourier series first appeared in the famous Sammlung Göschen in 1930. Although the Lebesgue theory is not used, the field covered is extensive and the treatment is thoroughly modern in outlook : the topic of convergence is not allowed to dominate the subject, and within the limits imposed the reader is introduced to most of the important concepts in the present-day theory. There are seven chapters : Introduction ; representation theory ; the completeness theorem ; convergence theory ; applications ; divergent Fourier series ; uniqueness theorems.

In itself an admirable guide to the theory, it forms, with the brilliant Hardy-Rogosinski Cambridge tract on the same subject, an excellent introduction to the more severe and detailed monographs on the modern theory of Fourier series, such as, for instance, the classic text of Zygmund.

T. A. A. B.

Analytical Geometry. By R. WALKER. Pp. 248. 12s. 6d. 1950. (Arnold)

I have no hesitation in commending this as a good and sound textbook ; I feel equally sure that a careful revision of a number of points of detail for the next edition (which will surely be required) can add considerably to its usefulness.

No previous knowledge of the subject is required, and the scope is intended to reach to the advanced (but not scholarship) level of the General Certificate of Education. Elementary calculus is used, and " determinants are occasionally mentioned, but no knowledge of them is necessary ". The chapter headings are Coordinates, Graphs and Loci, The Straight Line, Curves, Change of Axes, Equations representing Two Straight Lines, The Circle, Introduction to Conics, Curve Tracing. There is a liberal supply of worked examples, and a large number of examples for solution.

The work is clearly presented, with adequate diagrams to illustrate the text, and anyone using this book should acquire a sound grip of the essentials of the subject.

I entered a word of warning in my first paragraph, and must explain what I had in mind. A few examples may be given.

On p. 34, it is said that the equation, $xy =$ constant, " is known to represent a hyperbola with OL, OM for asymptotes ". Reference to the index supplies neither of the words *hyperbola, asymptotes*, and I see no earlier explanation. On p. 37 the phrase " approaches asymptotically " is used as if already familiar.

Example (iv) on p. 52 does not appear to illustrate the paragraph *The equation* $\lambda u + \lambda' u' = 0$ in which it is included ; example (ii), p. 71, seems to be incomplete as stated—attention to signs being required ; and in example (ii), p. 223, on the curve $(y-x)(y+2x) = x^3y^3$, the opening statement that, near the origin, " x and y are small and thus the term x^3y^3 is small in comparison with $(y-x)(y+2x)$ " seems dangerous without more explanation.

On p. 94, in discussing rotation of axes, the step

$$\begin{aligned} &" x = ON = \text{the projection of } OP \text{ on } OX \\ &\quad = \text{the projection of } ON' + \text{the projection of } N'P \\ &\quad = x' \cos\theta - y' \sin\theta " \end{aligned}$$

seems unsupported by previous discussion. At any rate, I could not find the reference about projection, and again the index did not help me.

Other points which seemed to require revision may be personal to me and not troublesome to anyone else ; but I found unsatisfying the introduction to tangency (p. 78), the introduction to conics (p. 153) and the reference to " imaginary " points (p. 193). I should also have liked the derivation of the " condition for two straight lines " in the text on p. 105 to have been scrapped and replaced by the worked example (iii) on p. 108.

Finally, I should add that the concentrated recital of apparent blemishes may give an air of disapproval which I by no means wish to convey. Any able teacher will be able to deal with these points without difficulty. But I do feel that life in the classroom may be made easier by an examination of the details of this book whenever the opportunity occurs. E. A. MAXWELL.

Matrizen : eine Darstellung für Ingenieure. By Dr.-Ing. R. ZURMÜHL. Pp. v-xv, 1–427. Dm. 25.50. 1950. (Springer, Berlin)

This is a book somewhat resembling Frazer, Duncan and Collar's *Elementary Matrices* (Cambridge, 1938), and having a similar purpose. It is closer, however, in its first five chapters, to a standard textbook on matrix algebra. These give an orthodox development, in great detail and with much care in exposition, of matrix notation, vectors, determinants and linear equations, bilinear and quadratic forms, symmetric, Hermitian, orthogonal, unitary and other special types of matrices, structure, canonical forms, the Segre and Weyr characteristics, invariant factors. Matrix functions with scalar parameters, infinite series in a matrix, these are also touched on. This part, on which an excellent general course in matrices might be based, occupies some 250 pages.

The remaining two chapters are quite long. Chapter VI is devoted to numerical methods, the solution of simultaneous linear equations and the approximation by iterative methods to characteristic roots and vectors, in both the real and the complex cases. This is well done, the illustrative examples being skilfully chosen. Methods due to Collatz, Wielandt, Hessenberg, Frazer, Duncan and Collar are described. A criticism might be that, as Dr. E. Bodewig has pointed out in a series of papers and reviews, the methods of solving linear equations associated with the names Banachiewicz and Cholesky really go back to Gauss.

Chapter VII is concerned with applications to networks, oscillations, systems of linear differential equations, matrix differentiation and integration, tensors, and estimation by least squares, in which the notation of Gauss and matrix notation are set out together.

A few omissions may be noticed. A proof of the rule for the determinant of a product of rectangular matrices (of orders m-by-n and n-by-m) and some reference to the notion of compound matrices based upon it might have made it possible to treat rank, and the unredundant conditions for positive definiteness, without the necessity of referring the reader to Bôcher.

Such criticisms are minor ; the book can be highly recommended. A. C. A.

Praktische Funktionenlehre. I. 2nd edition. By F. TOLKE. Pp. xi, 440. DM. 39. 1950. (Springer, Berlin)

Professor Tölke's massive compendium again exemplifies the thoroughness of German scholarship and the high standard of mathematics to which the German engineers and physicists (or some of them) are expected to attain. The volume falls roughly into two parts. In the first part, the theory of the elementary transcendental functions is collected and tabulated : the exponential properly comes first, as the function satisfying the integral equation

$$w(z) - w_0 - \int_0^z \omega w(\zeta) d\zeta = 0.$$

Then we have the logarithm, the general power, the trigonometric and hyperbolic functions, and their combinations, with a multitude of formulae, chiefly though not entirely on the algebraic side, comprising some 70 pages. Next follows about 90 pages of integrals, with integrands which are either algebraic or transcendental of the above type. The concluding section of this first part gives numerical tables occupying about 100 pages : the main contents are

a table of $2\pi x$, $\exp(\pm 2\pi x)$, $\sinh 2\pi x$,
$\cosh 2\pi x$, $\tanh 2\pi x$, $\coth 2\pi x$, $\sin 2\pi x$, $\cos 2\pi x$,
$\tan 2\pi x$, $\cot 2\pi x$, $\sin 2\pi(x - \frac{1}{4}\pi)$, $\cos 2\pi(x - \frac{1}{4}\pi)$,
$\tan 2\pi(x - \frac{1}{4}\pi)$, $\cot 2\pi(x - \frac{1}{4}\pi)$, $x = 0\ (0{\cdot}001)\ 1$, 5 decimals ;

a table of $\frac{1}{2}\pi x$, $\exp(\pm \frac{1}{2}\pi x)$, $\sin \frac{1}{2}\pi x$, $\cos \frac{1}{2}\pi x$, $x = 0\ (0{\cdot}01)\ 40$, 4 decimals ;

a table of $\mathrm{Ei}(\pm x)$, $\mathrm{Si}(x)$, $\mathrm{Ci}(x)$, $\mathrm{Si}(x)$, $\mathrm{Ci}(x)$, $x = 0\ (0{\cdot}01)\ 5$, 4 figures.

The second part is concerned with the application of the results listed in the first part to problems in applied mathematics, engineering, mathematical physics, such as, for instance, those which are encountered in the study of heat, hydraulics, elasticity. These can usually be formulated in terms of partial differential equations or in terms of integral equations, and although it is generally only possible to cope with somewhat simplified versions of real-life queries, these are dealt with in a way which will permit the final numerical results to be obtained to a fair degree of accuracy without going outside the covers of the book.

Of all such books, not simply of this one in particular, it is possible at times to feel that they are almost too encyclopaedic, that the inquiring engineer may be overwhelmed by the mass of detail before he learns to find his way about : this is, of course, not to disparage the value of these monumental efforts, but merely to suggest to the user that he should be thoroughly acquainted with a clear plan of the wood before he begins to look for individual trees. For the amazing care and industry, the patience and thoroughness of the author there can be nothing but praise. From the firm of Springer we expect the best in production and printing, and I should doubt whether

they have ever done better work than in producing with admirable clarity the multitudinous and complicated detail of this beautifully printed volume.

T. A. A. B.

Integraltafel. II. Bestimmte Integrale. By W. GRÖBNER and N. HOFREITER. Pp. vi, 204. 41s. 1950. (Springer, Vienna)

The first part of this compendium dealt with the indefinite integral and was reviewed in the *Gazette*, Vol. XXXIII, p. 149. The second part contains about 2500 formulae, very neatly reproduced from manuscript. After a short 5-page introduction on notation, on some methods of evaluating definite integrals, and a few references, the main work falls into five sections : rational integrands, with a sub-section on orthogonal polynomials (Legendre, Jacobi, Tchebichev, Laguerre, Hermite), irrational algebraic integrands, elementary transcendental integrands, the Gamma function and related integrals, integrals involving the cylinder (Bessel) functions. The pair of volumes forms a notable work of reference, containing in a reasonably accessible and identifiable form practically every integral that the average working mathematician is likely to need.

T. A. A. B.

Vorlesungen über höhere Mathematik. II. By A. DUSCHEK. Pp. vi, 386. 44s. 6d. ; geb. 49s. 1950. (Springer, Vienna)

The second volume of Duschek's admirable book takes the student of mathematics for physics and technology very nearly up to the end of his degree course : that is, it covers much the same ground as the pure mathematics of Schedule B of the joint report issued by the Institute of Physics and the Mathematical Association, the main exception being differential equations, which Duschek reserves for comprehensive treatment in his third volume. The first volume was reviewed in the *Gazette*, XXXIV, p. 145 : we need therefore add little about the plan of the whole work, save to remind readers that it is concerned with the mathematics itself, leaving the physical applications to be superimposed on the firm groundwork here supplied.

The first chapter deals with infinite series, giving the essentials of the matter without the trimmings, charming in themselves but inappropriate to the readers Duschek has in view. One section of this chapter deals with Fourier series, in which the function is restricted to have a piecewise continuous derivative ; this restriction, as is now fairly generally recognised, enables a valid demonstration to be given without using the second mean value theorem, while it is wide enough to embrace the main types of function the technologist is likely to encounter. Gibbs' phenomenon is briefly mentioned.

About half the book deals with functions of more than one real variable, derivatives, Taylor's theorem, maxima and minima, some geometrical applications, curvilinear, double and triple integrals and applications. All these topics are treated carefully and efficiently : students usually find this work heavy going, but Duschek's account is lucid and probably as easy to follow as the subject-matter will allow. Difficulties are fairly faced, not concealed : for example, the well-known Schwarz example to show the inadequacy of the " natural " definition of surface area is included.

As a sign of the times, probability receives good attention. There is a long chapter in the first volume on foundations, and the second volume contains another 50 pages on this topic. The Gaussian law of errors, the basic theorems of Bernoulli, Poisson and Bayes, the theory of errors and the method of least squares suffice to carry the subject far enough for those who are not obliged to have a specialist's knowledge.

Finally and most acceptably, there are 70 pages on determinants, matrices, linear algebra, vectors, tensor algebra and the elements of tensor analysis.

(Of course, determinants of orders two and three have already been freely used, and the theory is now general.) This sort of work lends itself to, perhaps even demands, crisp concise exposition ; but the abstract concepts are not always readily comprehended by the technical student, and Duschek's account is, I think, a little too austere. Perhaps in the second edition, which seems sure to be required in due course, the author will help the reader to grasp this very important set of ideas by adding more illustrative matter, particularly in the early stages of dealing with linear transformations, vectors and tensors, where quite simple geometrical examples can be of considerable assistance.

Exercises for the reader are given throughout : possibly a rather larger number could have been provided with advantage, but those we have are carefully chosen, and answers with useful hints on method are given at the end of the book.

The printing and format are in the usual first-rate Springer style.

T. A. A. B.

A Theory of Cross-spaces. By R. SCHATTEN. Pp. vi, 153. 16s. 1950. *Annals of Mathematics* Studies, 26. (Princeton University Press ; Geoffrey Cumberlege, Oxford University Press)

This work collects into one volume the results of recent researches mainly due to J. von Neumann and the author, on products of Banach spaces. Given two Banach spaces B_1 and B_2, $B_1 \cdot B_2$ may be defined as the set of formal finite sums

$$\sum_{i=1}^{n} f_i \times g_i, \quad f_i \in B_1, \quad g_i \in B_2,$$

where two such sums are to be identified if, for example, one may be obtained from the other by the application of the distributive law. It is then possible —in fact, in many different ways—to assign a norm to $B_1 \cdot B_2$. If the norm, say α, is, in a natural sense, consistent with the norms in B_1 and B_2, it is called a cross-norm and $B_1 \cdot B_2$ furnished with this cross-norm is written $B_1 \cdot_\alpha B_2$. As a topological space, this latter is not, in general, complete. If it is completed by the usual Cantor process of replacing elements by fundamental sequences, the resulting Banach space, written $B_1 \times_\gamma B_2$, is called the cross-space of B_1 and B_2 with respect to the norm α.

After a preliminary survey of the problem and the results obtained, the author devotes the first four chapters to a general discussion of the properties of cross-norms and cross-spaces and the characterisation of cross-spaces as spaces of linear operators. If B_1^*, B_2^* are the spaces conjugate to B_1, B_2, then the set $B_1^* \cdot B_2^*$ may be defined, together with an " inner product " between $B_1 \cdot B_2$ and $B_1^* \cdot B_2^*$. Moreover, if α is a norm on $B_1 \cdot B_2$, an associate norm α' may be defined on $B_1^* \cdot B_2^*$, such that

$$(B_1 \times_\alpha B_2)^* \supset B_1^* \times_{\alpha'} B_2^*$$

(the latter space being called the associate space of $B_1 \times_\alpha B_2$). Among the cross-norms on $B_1 \cdot B_2$, there are always two, γ and λ, such that γ is the greatest cross-norm and λ is the least cross-norm whose associate is also a cross-norm, (a least cross-norm does not always exist). In fact, interpreting $\Sigma f_i \times g_i$ as the operator A, of finite rank, from B_1^* to B_2 given by

$$A(F) = \sum_{i=1}^{n} F(f_i) g_i, \qquad F \in B_1^*,$$

λ is simply the bound of the operator A. Moreover, λ and γ have a certain

" uniformity " property, and $\lambda' = \lambda$ (as a norm on $B_1^* \cdot B_2^*$). A characterisation of $(B_1 \times_\alpha B_2)^*$ and its subspace $B_1^* \times_{\alpha'} B_2^*$ as certain Banach spaces of operators from B_1 to B_2^* is given.

In Chapter V the theory is applied to crossed unitary spaces (cross-spaces of complete unitary spaces) and thus, in particular, to finite euclidean and Hilbert spaces. If H is a complete unitary space, and H^* the space of additive bounded functionals on H, it is shown, for example, that λ is the least uniform cross-norm on $H \cdot H^*$, by establishing the coincidence of the class of uniform cross-norms with a certain class of " unitarily invariant " cross-norms. Another result specially applicable to $H \cdot H^*$ is that $\alpha'' = \alpha$ for any uniform cross-norm α (in general $\alpha'' \leqslant \alpha$ and $\alpha''' = \alpha'$, where $B_1^{**} \cdot B_2^{**}$ may be regarded, in an obvious way, as containing $B_1 \cdot B_2$). An interpretation of $H \times_\sigma H$ as a certain class (Schmidt-class) of operators on H, for a particular cross-norm σ, is given. There are two short appendices.

It is unfortunate that the author has been unable to make his treatment more self-contained. The dependence of the first four chapters on a familiarity with the contents of Banach's *Théorie des Opérations linéaires* was probably inevitable, but it does seem that the treatment in Chapter V is somewhat abrupt and requires familiarity with, for example, M. H. Stone's *Linear Transformations in Hilbert Space* and some of von Neumann's publications for an understanding not only of the proofs, but even of some of the statements and notations. However, a very full bibliography compensates for this dependence.

Misprints are avoidably but not annoyingly frequent; and the use of a characteristically German word order (" of all converging to 0 sequences," . . . " an inner product has been added in the described above manner ") reflects on the style rather than on the lucidity of the text, and certainly does not sensibly detract from this book's great value to the specialist in the field.

P. J. H.

Analyse mathématique. Par P. APPELL. 6th edition, revised by G. VALIRON. Pp. ix, 408: 434. 2000 fr.; 2200 fr. 1951. (Gauthier-Villars)

The first edition of this book appeared in 1898, and was based on Appell's lectures at the great college of industrial and commercial technology, the École Centrale des Arts et Manufactures, in Paris. Successive editions incorporated the substance of lectures given by Appell at the Sorbonne, mainly on analysis, though the first three chapters, which deal with the elements of coordinate geometry in the plane and in space, formed an introduction to the mechanics course. In the French system, this is a " Cours de mathématiques générales ", and roughly it corresponds to the non-specialist part of an Honours Degree course in mathematics in this country. On the whole, its appeal is to those who will require a high standard of mathematical technique for application to other studies, though at the same time matters which are passed over rapidly or assumed as known in the classic French treatises on analysis are here dealt with in full detail.

After the geometrical chapters mentioned above, the first volume proceeds to the concepts of limits, continuity, the derivative and integral of functions of a single real variable, including a study of the exponential and logarithmic functions, based here on an investigation of the meaning of a^x, first for rational x, and then for irrational x as the limit of sequences of rationals. After a long chapter on methods of indefinite integration and one on infinite integrals, we have an account of functions of two variables, and then the first volume concludes with sections on curves, plane and twisted, curvilinear integrals and the calculation of areas and volumes.

The second volume starts with differential equations of the first and second

orders, with a short illustrative section on some simple partial differential equations. Then we have Taylor's theorem and some of its applications, and a long chapter on infinite series and expansions of functions in power series and in Fourier series ; in the latter section the emphasis is on the applicability of the Fourier technique, and so effectively all that is proved is that a continuous function will be the sum of its Fourier series provided the latter is normally convergent (that is, satisfies Weierstrass's M-test). The appearance next of a chapter on complex number is explained by M. Valiron in the preface, but, even so, such a belated account involves a certain amount of cumbersome manipulation in earlier stages. Two solid and clearly-written chapters deal with double and triple integrals and some dynamical applications. The last two chapters are new to this edition. As a sign of the times, one is on probability, a short account which would, of course, need supplementing, since it claims to be only an introduction. The other new chapter is on determinants, including something about quadratic forms, linear substitutions and matrices ; it is a little surprising to find that the book had arrived at a sixth edition before feeling a need for this topic, if only as a convenient shorthand.

There is a short appendix on planimeters, and a collection of nearly 400 exercises. It is, of course, superfluous to add that the printing is clear and pleasing in the style for which Gauthier-Villars have long been renowned.

Only a French mathematician could pronounce judgment on nuances of style. To the foreigner it is remarkable that the considerable number of French treatises on analysis, differing so much in character, should appear so similar in style—clear, precise and fluent. It may be that French is a language particularly well suited to mathematical exposition ; but I suspect that the uniform standard of excellence is partly due to the fact that Frenchmen are as careful and as proud of their language as they are of their mathematics. T. A. A. B.

. **Problèmes de Propagation Guidées des Ondes Electromagnétiques.** Par L. DE BROGLIE. Pp. vi, 118. 1100 fr. 1951. (Gauthier-Villars, Paris)

The second edition of this work differs little from the edition of 1940. Although the author in his preface expresses the hope that this book will be of use to radio engineers, his approach to the subject is that of a mathematician rather than that of a physicist. There is no reference to experimental confirmation of the results derived : indeed the book might well have been written before any experimental work on the subject had been done. This limitation enables the author to give a very concise and lucid account of some of the more elementary theory of wave propagation in confined spaces.

Starting from Maxwell's equations in rectangular coordinates the author develops potential functions which enable him to find with the minimum of effort the electromagnetic fields in guides and cavities with perfectly conducting walls of various shapes. After dealing briefly with the attenuation which occurs in guides with walls of finite conductivity, an approach is made to the difficult subject of wave propagation in horns. In the last chapter Huyghen's principle is discussed in the light of Maxwell's equations, and Kottler's formulae applied to some problems in diffraction.

The book's chief defects are its lack of diagrams showing the fields in the cases considered, and the omission (which the author regrets in the preface) of recent British and American references. J. C. S. RICHARDS.

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